SETS AND COUNTING

2

Recently, 1,000 college seniors were asked whether they favored increasing their state's gasoline tax to generate funds to improve highways and whether they favored increasing their state's alcohol tax to generate funds to improve the public education system. The responses were tallied, and the following results were printed in the campus newspaper: 746 students favored an increase in the gasoline tax, 602 favored an increase in the alcohol tax, and 449 favored increase in both taxes. How many of these 1,000 students favored an increase in at least one of the taxes? How many favored increasing only the gasoline tax? How many favored increasing only the alcohol tax? How many favored increasing neither tax?

The mathematical tool that was designed to answer questions like these is

continued

WHAT WE WILL DO IN THIS CHAPTER

WE'LL USE VENN DIAGRAMS TO DEPICT THE RELATIONSHIPS BETWEEN SETS:

- One set might be contained within another set.
- Two or more sets might, or might not, share elements in common.

WE'LL EXPLORE APPLICATIONS OF VENN DIAGRAMS:

- The results of consumer surveys, marketing analyses, and political polls can be analyzed by using Venn diagrams.
- Venn diagrams can be used to prove general formulas related to set theory.

WE'LL EXPLORE VARIOUS METHODS OF COUNTING:

- A fundamental principle of counting is used to determine the total number of possible ways of selecting specified items. For example, how many different student ID numbers are possible at your school?

continued

WHAT WE WILL DO IN THIS CHAPTER — *continued*

- In selecting items from a specified group, sometimes the order in which the items are selected matters (the awarding of prizes: first, second, and third), and sometimes it does not (selecting numbers in a lottery or people for a committee). How does this affect your method of counting?

WE'LL USE SETS IN VARIOUS CONTEXTS:

- In this text, we will use set theory extensively in Chapter 3 on probability.
- Many standardized admissions tests, such as the Graduate Record Exam (GRE) and the Law School Admissions Test (LSAT), ask questions that can be answered with set theory.

WE'LL EXPLORE SETS THAT HAVE AN INFINITE NUMBER OF ELEMENTS:

- One-to-one correspondences are used to "count" and compare the number of elements in infinite sets.
- Not all infinite sets have the same number of elements; some infinite sets are countable, and some are not.

the *set. Webster's New World College Dictionary* defines a **set** as "a prescribed collection of points, numbers, or other objects that satisfy a given condition." Although you might be able to answer the questions about taxes without any formal knowledge of sets, the mental reasoning involved in obtaining your answers uses some of the basic principles of sets. (Incidentally, the answers to the above questions are 899, 297, 153, and 101, respectively.)

The branch of mathematics that deals with sets is called **set theory.** Set theory can be helpful in solving both mathematical and nonmathematical problems. We will explore set theory in the first half of this chapter. As the above example shows, set theory often involves the analysis of the relationships between sets and counting the number of elements in a specific category. Consequently, various methods of counting, collectively known as **combinatorics,** will be developed and discussed in the second half of this chapter. Finally, what if a set has too many elements to count by using finite numbers? For example, how many integers are there? How many real numbers? The chapter concludes with an exploration of infinite sets and various "levels of infinity."

2.1 Sets and Set Operations

Objectives

- Learn the basic vocabulary and notation of set theory
- Learn and apply the union, intersection, and complement operations
- Draw Venn diagrams

A **set** is a collection of objects or things. The objects or things in the set are called **elements** (or *members*) of the set. In our example above, we could talk about the *set* of students who favor increasing only the gasoline tax or the *set* of students who do not favor increasing either tax. In geography, we can talk about the *set* of all state capitals or the *set* of all states west of the Mississippi. It is easy to determine whether something is in these sets; for example, Des Moines is an element of the set of state capitals, whereas Dallas is not. Such sets are called **well-defined** because there is a way of determining for sure whether a particular item is an element of the set.

EXAMPLE 1

DETERMINING WELL-DEFINED SETS Which of the following sets are well-defined?

a. the set of all movies directed by Alfred Hitchcock
b. the set of all great rock-and-roll bands
c. the set of all possible two-person committees selected from a group of five people

SOLUTION

a. This set is well-defined; either a movie was directed by Hitchcock, or it was not.
b. This set is *not* well-defined; membership is a matter of opinion. Some people would say that the Ramones (one of the pioneer punk bands of the late 1970s) are a member, while others might say they are not. (Note: The Ramones were inducted into the Rock and Roll Hall of Fame in 2002.)
c. This set is well-defined; either the two people are from the group of five, or they are not.

Notation

By tradition, a set is denoted by a capital letter, frequently one that will serve as a reminder of the contents of the set. **Roster notation** (also called *listing notation*) is a method of describing a set by listing each element of the set inside the symbols { and }, which are called *set braces*. In a listing of the elements of a set, each distinct element is listed only once, and the order of the elements doesn't matter.

The symbol $\in$ stands for the phrase *is an element of,* and $\notin$ stands for *is not an element of*. The **cardinal number** of a set A is the number of elements in the set and is denoted by $n(A)$. Thus, if R is the set of all letters in the name "Ramones," then $R = \{\text{r, a, m, o, n, e, s}\}$. Notice that m is an element of the set R, x is not an element of R, and R has 7 elements. In symbols, $\text{m} \in R$, $\text{x} \notin R$, and $n(R) = 7$.


Roberta Bayley/Evening Standard/Hulton Archive/Getty Images

The "Ramones" or The "Moaners"? The set R of all letters in the name "Ramones" is the same as the set M of all letters in the name "Moaners." Consequently, the sets are equal; $M = R = \{a, e, m, n, o, r, s\}$. (R.I.P. Joey Ramone 1951–2001, Dee Dee Ramone 1952–2002, Johnny Ramone 1948–2004.)

Two sets are **equal** if they contain exactly the same elements. *The order in which the elements are listed does not matter.* If M is the set of all letters in the name "Moaners," then $M = \{m, o, a, n, e, r, s\}$. This set contains exactly the same elements as the set R of letters in the name "Ramones." Therefore, $M = R = \{a, e, m, n, o, r, s\}$.

Often, it is not appropriate or not possible to describe a set in roster notation. For extremely large sets, such as the set V of all registered voters in Detroit, or for sets that contain an infinite number of elements, such as the set G of all negative real numbers, the roster method would be either too cumbersome or impossible to use. Although V could be expressed via the roster method (since each county compiles a list of all registered voters in its jurisdiction), it would take hundreds or even thousands of pages to list everyone who is registered to vote in Detroit! In the case of the set G of all negative real numbers, no list, no matter how long, is capable of listing all members of the set; there is an infinite number of negative numbers.

In such cases, it is often necessary, or at least more convenient, to use **set-builder notation,** which lists the rules that determine whether an object is an element of the set rather than the actual elements. A set-builder description of set G above is

$$G = \{x \mid x < 0 \quad \text{and} \quad x \in \Re\}$$

which is read as "the set of all x such that x is less than zero and x is a real number." A set-builder description of set V above is

$$V = \{\text{persons} \mid \text{the person is a registered voter in Detroit}\}$$

which is read as "the set of all persons such that the person is a registered voter in Detroit." In set-builder notation, the vertical line stands for the phrase "such that."

Whatever is on the left side of the line is the general type of thing in the set, while the rules about set membership are listed on the right.

EXAMPLE 2

READING SET-BUILDER NOTATION Describe each of the following in words.

a. $\{x \mid x > 0 \text{ and } x \in \Re\}$
b. {persons | the person is a living former U.S. president}
c. {women | the woman is a former U.S. president}

SOLUTION

a. the set of all x such that x is a positive real number
b. the set of all people such that the person is a living former U.S. president
c. the set of all women such that the woman is a former U.S. president

The set listed in part (c) of Example 2 has no elements; there are no women who are former U.S. presidents. If we let W equal "the set of all women such that the woman is a former U.S. president," then $n(W) = 0$. A set that has no elements is called an **empty set** and is denoted by $\emptyset$ or by $\{\}$. Notice that since the empty set has no elements, $n(\emptyset) = 0$. In contrast, the set $\{0\}$ is not empty; it has one element, the number zero, so $n(\{0\}) = 1$.

Universal Set and Subsets

When we work with sets, we must define a universal set. For any given problem, the **universal set**, denoted by U, is the set of all possible elements of any set used in the problem. For example, when we spell words, U is the set of all letters in the alphabet. When every element of one set is also a member of another set, we say that the first set is a *subset* of the second; for instance, {p, i, n} is a subset of {p, i, n, e}. In general, we say that A is a **subset** of B, denoted by $A \subseteq B$, if for every $x \in A$ it follows that $x \in B$. Alternatively, $A \subseteq B$ if A contains no elements that are not in B. If A contains an element that is not in B, then A is not a subset of B (symbolized as $A \not\subseteq B$).

EXAMPLE 3

DETERMINING SUBSETS Let B = {countries | the country has a permanent seat on the U.N. Security Council}. Determine whether A is a subset of B.

a. A = {Russian Federation, United States}
b. A = {China, Japan}
c. A = {United States, France, China, United Kingdom, Russian Federation}
d. $A = \{\ \}$

SOLUTION

We use the roster method to list the elements of set B.

$$B = \{\text{China, France, Russian Federation, United Kingdom, United States}\}$$

a. Since every element of A is also an element of B, A is a subset of B; $A \subseteq B$.
b. Since A contains an element (Japan) that is not in B, A is not a subset of B; $A \not\subseteq B$.
c. Since every element of A is also an element of B (note that $A = B$), A is a subset of B (and B is a subset of A); $A \subseteq B$ (and $B \subseteq A$). In general, every set is a subset of itself; $A \subseteq A$ for any set A.
d. Does A contain an element that is not in B? No! Therefore, A (an empty set) is a subset of B; $A \subseteq B$. In general, the empty set is a subset of all sets; $\emptyset \subseteq A$ for any set A.

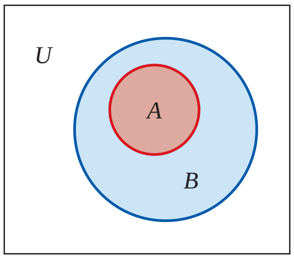


FIGURE 2.1
A is a subset of *B*. $A \subseteq B$.

We can express the relationship $A \subseteq B$ visually by drawing a Venn diagram, as shown in Figure 2.1. A **Venn diagram** consists of a rectangle, representing the universal set, and various closed figures within the rectangle, each representing a set. Recall that Venn diagrams were used in Section 1.1 to determine whether an argument was valid.

If two sets are equal, they contain exactly the same elements. It then follows that each is a subset of the other. For example, if $A = B$, then every element of *A* is an element of *B* (and vice versa). In this case, *A* is called an **improper subset** of *B*. (Likewise, *B* is an improper subset of *A*.) Every set is an improper subset of itself; for example, $A \subseteq A$. On the other hand, if *A* is a subset of *B* and *B* contains an element not in *A* (that is, $A \neq B$), then *A* is called a **proper subset** of *B*. To indicate a proper subset, the symbol $\subset$ is used. While it is acceptable to write $\{1, 2\} \subseteq \{1, 2, 3\}$, the relationship of a proper subset is stressed when it is written $\{1, 2\} \subset \{1, 2, 3\}$. Notice the similarities between the subset symbols, $\subset$ and $\subseteq$, and the inequality symbols, $<$ and $\leq$, used in algebra; it is acceptable to write $1 \leq 3$, but writing $1 < 3$ is more informative.

Intersection of Sets

Sometimes an element of one set is also an element of another set; that is, the sets may overlap. This overlap is called the **intersection** of the sets. If an element is in two sets *at the same time*, it is in the intersection of the sets.

INTERSECTION OF SETS

The **intersection** of set *A* and set *B*, denoted by $A \cap B$, is

$$A \cap B = \{x \mid x \in A \quad \text{and} \quad x \in B\}$$

The intersection of two sets consists of those elements that are common to both sets.

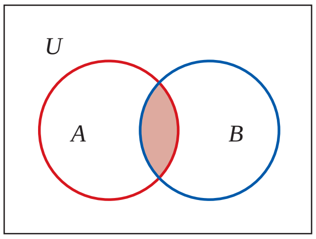


FIGURE 2.2
The intersection $A \cap B$ is represented by the (overlapping) shaded region.

For example, given the sets $A = \{\text{Buffy, Spike, Willow, Xander}\}$ and $B = \{\text{Angel, Anya, Buffy, Giles, Spike}\}$, their intersection is $A \cap B = \{\text{Buffy, Spike}\}$.

Venn diagrams are useful in depicting the relationship between sets. The Venn diagram in Figure 2.2 illustrates the intersection of two sets; the shaded region represents $A \cap B$.

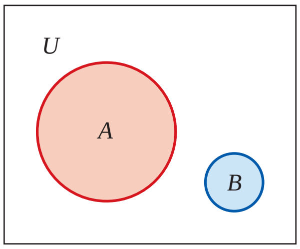


FIGURE 2.3
Mutually exclusive sets have no elements in common ($A \cap B = \emptyset$).

Mutually Exclusive Sets

Sometimes a pair of sets has no overlap. Consider an ordinary deck of playing cards. Let $D = \{\text{cards} \mid \text{the card is a diamond}\}$ and $S = \{\text{cards} \mid \text{the card is a spade}\}$. Certainly, *no* cards are both diamonds and spades *at the same time;* that is, $S \cap D = \emptyset$.

Two sets *A* and *B* are **mutually exclusive** (or *disjoint*) if they have no elements in common, that is, if $A \cap B = \emptyset$. The Venn diagram in Figure 2.3 illustrates mutually exclusive sets.

Union of Sets

What does it mean when we ask, "How many of the 500 college students in a transportation survey own an automobile or a motorcycle?" Does it mean "How many students own either an automobile or a motorcycle *or both*?" or does it mean "How many students own either an automobile or a motorcycle, *but not both*?" The former is called the *inclusive or,* because it includes the possibility of owning both; the latter is called the *exclusive or.* In logic and in mathematics, the word *or* refers to the *inclusive or,* unless you are told otherwise.

The meaning of the word *or* is important to the concept of union. The **union** of two sets is a new set formed by joining those two sets together, just as the union of the states is the joining together of fifty states to form one nation.

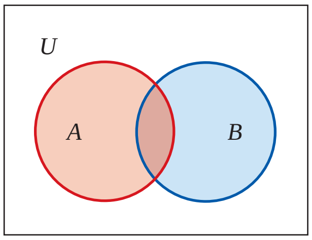


FIGURE 2.4

The union $A \cup B$ is represented by the (entire) shaded region.

UNION OF SETS

The **union** of set A and set B, denoted by $A \cup B$, is

$$A \cup B = \{x \mid x \in A \quad \text{or} \quad x \in B\}$$

The union of A and B consists of all elements that are in either A or B or both, that is, all elements that are in at least one of the sets.

For example, given the sets A = {Conan, David} and B = {Ellen, Katie, Oprah}, their union is $A \cup B$ = {Conan, David, Ellen, Katie, Oprah}, and their intersection is $A \cap B = \varnothing$. Note that because they have no elements in common, A and B are mutually exclusive sets. The Venn diagram in Figure 2.4 illustrates the union of two sets; the entire shaded region represents $A \cup B$.

EXAMPLE 4

FINDING THE INTERSECTION AND UNION OF SETS Given the sets $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, find the following.

a. $A \cap B$ (the intersection of A and B)
b. $A \cup B$ (the union of A and B)

SOLUTION

a. The intersection of two sets consists of those elements that are common to both sets; therefore, we have

$$\begin{aligned} A \cap B &= \{1, 2, 3\} \cap \{2, 4, 6\} \\ &= \{2\} \end{aligned}$$

b. The union of two sets consists of all elements that are in at least one of the sets; therefore, we have

$$\begin{aligned} A \cup B &= \{1, 2, 3\} \cup \{2, 4, 6\} \\ &= \{1, 2, 3, 4, 6\} \end{aligned}$$

The Venn diagram in Figure 2.5 shows the composition of each set and illustrates the intersection and union of the two sets.

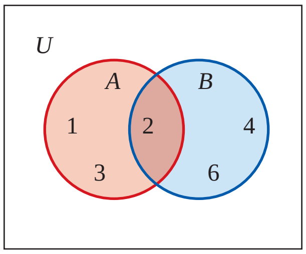


FIGURE 2.5

The composition of sets A and B in Example 4.

Because $A \cup B$ consists of all elements that are in A or B (or both), to find $n(A \cup B)$, we add $n(A)$ plus $n(B)$. However, doing so results in an answer that might be too big; that is, if A and B have elements in common, these elements will be counted twice (once as a part of A and once as a part of B). Therefore, to find the cardinal number of $A \cup B$, we add the cardinal number of A to the cardinal number of B and then *subtract* the cardinal number of $A \cap B$ (so that the overlap is not counted twice).

CARDINAL NUMBER FORMULA FOR THE UNION/INTERSECTION OF SETS

For any two sets A and B, the number of elements in their union is $n(A \cup B)$, where

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and $n(A \cap B)$ is the number of elements in their intersection.

As long as any three of the four quantities in the general formula are known, the missing quantity can be found by algebraic manipulation.

EXAMPLE 5

ANALYZING THE COMPOSITION OF A UNIVERSAL SET Given $n(U) = 169$, $n(A) = 81$, and $n(B) = 66$, find the following.

a. If $n(A \cap B) = 47$, find $n(A \cup B)$ and draw a Venn diagram depicting the composition of the universal set.

b. If $n(A \cup B) = 147$, find $n(A \cap B)$ and draw a Venn diagram depicting the composition of the universal set.

SOLUTION

a. We must use the Union/Intersection Formula. Substituting the three given quantities, we have

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 81 + 66 - 47 \\ &= 100 \end{aligned}$$

The Venn diagram in Figure 2.6 illustrates the composition of U.

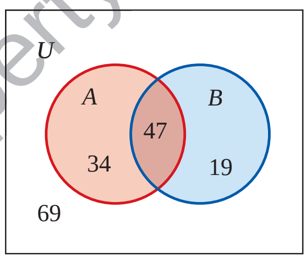


FIGURE 2.6 $n(A \cap B) = 47$.

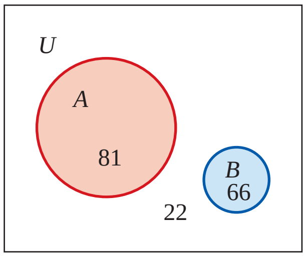


FIGURE 2.7 $n(A \cup B) = 147$.

b. We must use the Union/Intersection Formula. Substituting the three given quantities, we have

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 147 &= 81 + 66 - n(A \cap B) \\ 147 &= 147 - n(A \cap B) \\ n(A \cap B) &= 147 - 147 \\ n(A \cap B) &= 0 \end{aligned}$$

Therefore, A and B have no elements in common; they are mutually exclusive. The Venn diagram in Figure 2.7 illustrates the composition of U.

EXAMPLE 6

ANALYZING THE RESULTS OF A SURVEY A recent transportation survey of 500 college students (the universal set U) yielded the following information: 291 students own an automobile (A), 179 own a motorcycle (M), and 85 own both an automobile and a motorcycle ($A \cap M$). What percent of these students own an automobile or a motorcycle?

SOLUTION

Recall that "automobile or motorcycle" means "automobile or motorcycle or both" (the inclusive *or*) and that *or* implies union. Hence, we must find $n(A \cup M)$, the cardinal number of the union of sets A and M. We are given that $n(A) = 291$, $n(M) = 179$, and $n(A \cap M) = 85$. Substituting the given values into the Union/Intersection Formula, we have

$$\begin{aligned} n(A \cup M) &= n(A) + n(M) - n(A \cap M) \\ &= 291 + 179 - 85 \\ &= 385 \end{aligned}$$

Therefore, 385 of the 500 students surveyed own an automobile or a motorcycle. Expressed as a percent, $385/500 = 0.77$; therefore, 77% of the students own an automobile or a motorcycle (or both).

Complement of a Set

In certain situations, it might be important to know how many things are *not* in a given set. For instance, when playing cards, you might want to know how many cards are not ranked lower than a five; or when taking a survey, you might want to know how many people did not vote for a specific proposition. The set of all elements in the universal set that are *not* in a specific set is called the *complement* of the set.

COMPLEMENT OF A SET

The **complement** of set A, denoted by A' (read "A prime" or "the complement of A"), is

$$A' = \{x \mid x \in U \quad \text{and} \quad x \notin A\}$$

The complement of a set consists of all elements that are in the universal set but not in the given set.

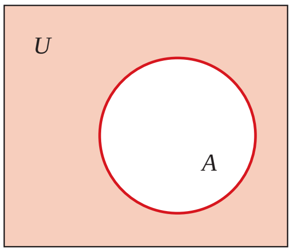


FIGURE 2.8

The complement A' is represented by the shaded region.

For example, given that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 7, 9\}$, the complement of A is $A' = \{2, 4, 6, 8\}$. What is the complement of A'? Just as $-(-x) = x$ in algebra, $(A')' = A$ in set theory. The Venn diagram in Figure 2.8 illustrates the complement of set A; the shaded region represents A'.

Suppose A is a set of elements, drawn from a universal set U. If x is an element of the universal set ($x \in U$), then exactly one of the following must be true: (1) x is an element of A ($x \in A$), or (2) x is not an element of A ($x \notin A$). Since no element of the universal set can be in both A and A' at the same time, it follows that A and A' are mutually exclusive sets whose union equals the entire universal set. Therefore, the sum of the cardinal numbers of A and A' equals the cardinal number of U.

HISTORICAL NOTE

JOHN VENN, 1834–1923

Courtesy The Masters and Fellows of Gonville and Caius College, Cambridge

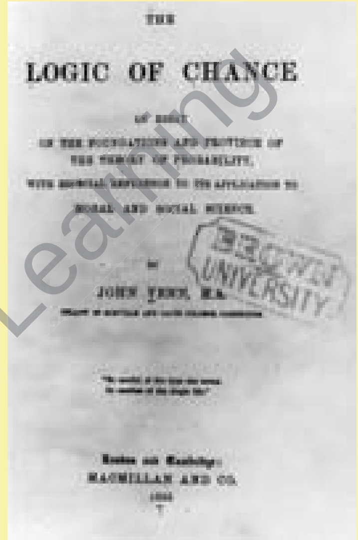
John Venn is considered by many to be one of the originators of modern symbolic logic. Venn received his degree in mathematics from the University at Cambridge at the age of twenty-three. He was then elected a fellow of the college and held this fellowship until his death, some 66 years later. Two years after receiving his degree, Venn accepted a teaching position at Cambridge: college lecturer in moral sciences.

During the latter half of the nineteenth century, the study of logic experienced a rebirth in England. Mathematicians were attempting to symbolize and quantify the central concepts of logical thought. Consequently, Venn chose to focus on the study of logic during his tenure at Cambridge. In addition, he investigated the field of probability and published *The Logic of Chance,* his first major work, in 1866.

Venn was well read in the works of his predecessors, including the noted logicians Augustus De Morgan, George Boole, and Charles Dodgson (a.k.a. Lewis Carroll). Boole's pioneering work on the marriage of logic and algebra proved to be a strong influence on Venn; in fact, Venn used the type of diagram that now bears his name in an 1876 paper in which he examined Boole's system of symbolic logic.

Venn was not the first scholar to use the diagrams that now bear his name. Gottfried Leibniz, Leonhard Euler, and others utilized similar diagrams years before Venn did. Examining each author's diagrams, Venn was critical of their lack of uniformity. He developed a consistent, systematic explanation of the general use of geometrical figures in the analysis of logical arguments. Today, these geometrical figures are known by his name and are used extensively in elementary set theory and logic.

Venn's writings were held in high esteem. His textbooks, *Symbolic Logic* (1881) and *The Principles of Empirical Logic* (1889), were used during the late nineteenth and early twentieth centuries. In addition to his works on logic and probability, Venn conducted much research into historical records, especially those of his college and those of his family.


THE
LOGIC OF CHANCE
JOHN VENN, M.A.
MACMILLAN AND CO.
BROWN UNIVERSITY

Brown University Library

Set theory and the cardinal numbers of sets are used extensively in the study of probability. Although he was a professor of logic, Venn investigated the foundations and applications of theoretical probability. Venn's first major work, The Logic of Chance, *exhibited the diversity of his academic interests.*

It is often quicker to count the elements that are *not* in a set than to count those that are. Consequently, to find the cardinal number of a set, we can subtract the cardinal number of its complement from the cardinal number of the universal set; that is, $n(A) = n(U) - n(A')$.

CARDINAL NUMBER FORMULA FOR THE COMPLEMENT OF A SET

For any set A and its complement A',

$$n(A) + n(A') = n(U)$$

where U is the universal set.

Alternatively,

$$n(A) = n(U) - n(A') \qquad \text{and} \qquad n(A') = n(U) - n(A)$$

EXAMPLE 7

USING THE COMPLEMENT FORMULA How many letters in the alphabet precede the letter w?

SOLUTION

Rather than counting all the letters that precede w, we will take a shortcut by counting all the letters that do *not* precede w. Let $L = \{\text{letters} \mid \text{the letter precedes w}\}$. Therefore, $L' = \{\text{letter} \mid \text{the letter does not precede w}\}$. Now $L' = \{\text{w, x, y, z}\}$, and $n(L') = 4$; therefore, we have

$$\begin{aligned} n(L) &= n(U) - n(L') \quad \textbf{Complement Formula} \\ &= 26 - 4 \\ &= 22 \end{aligned}$$

There are twenty-two letters preceding the letter w.

Shading Venn Diagrams

In an effort to visualize the results of operations on sets, it may be necessary to shade specific regions of a Venn diagram. The following example shows a systematic method for shading the intersection or union of any two sets.

EXAMPLE 8

SHADING VENN DIAGRAMS On a Venn diagram, shade in the region corresponding to the indicated set.

a. $A \cap B'$ **b.** $A \cup B'$

SOLUTION

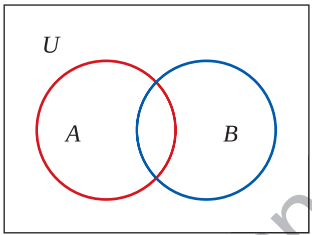


FIGURE 2.9
Two overlapping circles.

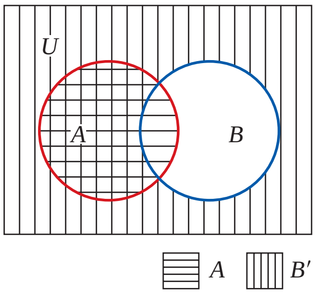
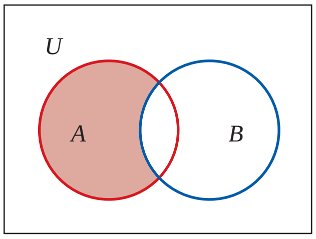


FIGURE 2.10

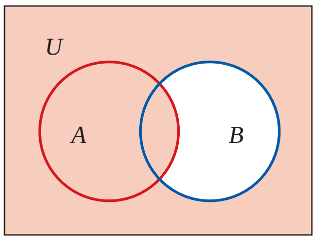
a. First, draw and label two overlapping circles as shown in Figure 2.9. The two "components" of the operation $A \cap B'$ are "A" and "B'." Shade each of these components in contrasting ways; shade one of them, say A, with horizontal lines, and the other with vertical lines as in Figure 2.10. Be sure to include a legend, or key, identifying each type of shading.

To be in the intersection of two sets, an element must be in *both* sets at the same time. Therefore, the intersection of A and B' is the region that is shaded in *both* directions (horizontal and vertical) at the same time. A final diagram depicting $A \cap B'$ is shown in Figure 2.11.



$A \cap B'$

FIGURE 2.11



$A \cup B'$

FIGURE 2.12

b. Refer to Figure 2.10. To be in the union of two sets, an element must be in *at least one* of the sets. Therefore, the union of A and B' consists of all regions that are shaded in *any* direction whatsoever (horizontal or vertical or both). A final diagram depicting $A \cup B'$ is shown in Figure 2.12.

Set Theory and Logic

If you have read Chapter 1, you have probably noticed that set theory and logic have many similarities. For instance, the union symbol $\cup$ and the disjunction symbol $\vee$ have the same meaning, but they are used in different circumstances; $\cup$ goes between sets, while $\vee$ goes between logical expressions. The $\cup$ and $\vee$ symbols are similar in appearance because their usages are similar. A comparison of the terms and symbols used in set theory and logic is given in Figure 2.13.

Set Theory		Logic		Common Wording
Term	**Symbol**	**Term**	**Symbol**	
union	$\cup$	disjunction	$\vee$	or
intersection	$\cap$	conjunction	$\wedge$	and
complement	$'$	negation	$\sim$	not
subset	$\subseteq$	conditional	$\rightarrow$	if . . . then . . .

FIGURE 2.13 Comparison of terms and symbols used in set theory and logic.

Applying the concepts and symbols of Chapter 1, we can define the basic operations of set theory in terms of logical biconditionals. The biconditionals in Figure 2.14 are tautologies (expressions that are always true); the first biconditional is read as "*x* is an element of the union of sets *A* and *B* if and only if *x* is an element of set *A* or *x* is an element of set *B*."

Basic Operations in Set Theory	Logical Biconditional
union	$[x \in (A \cup B)] \leftrightarrow [x \in A \vee x \in B]$
intersection	$[x \in (A \cap B)] \leftrightarrow [x \in A \wedge x \in B]$
complement	$(x \in A') \leftrightarrow \sim (x \in A)$
subset	$(A \subseteq B) \leftrightarrow (x \in A \rightarrow x \in B)$

FIGURE 2.14 Set theory operations as logical biconditionals.

2.1 Exercises

1. State whether the given set is well defined.
- **a.** the set of all black automobiles
- **b.** the set of all inexpensive automobiles
- **c.** the set of all prime numbers
- **d.** the set of all large numbers

2. Suppose $A = \{2, 5, 7, 9, 13, 25, 26\}$.
- **a.** Find $n(A)$
- **b.** True or false: $7 \in A$
- **c.** True or false: $9 \notin A$
- **d.** True or false: $20 \notin A$

In Exercises 3–6, list all subsets of the given set. Identify which subsets are proper and which are improper.

3. $B = \{\text{Lennon, McCartney}\}$

4. $N = \{0\}$

5. $S = \{\text{yes, no, undecided}\}$

6. $M = \{\text{classical, country, jazz, rock}\}$

In Exercises 7–10, the universal set is $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

7. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$, find the following.

a. $A \cap B$ **b.** $A \cup B$
c. A' **d.** B'

8. If $A = \{2, 3, 5, 7\}$ and $B = \{2, 4, 6, 7\}$, find the following.

a. $A \cap B$ **b.** $A \cup B$
c. A' **d.** B'

9. If $A = \{1, 3, 5, 7, 9\}$ and $B = \{0, 2, 4, 6, 8\}$, find the following.

a. $A \cap B$ **b.** $A \cup B$
c. A' **d.** B'

10. If $A = \{3, 6, 9\}$ and $B = \{4, 8\}$, find the following.

a. $A \cap B$ **b.** $A \cup B$
c. A' **d.** B'

In Exercises 11–16, the universal set is U = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}. If A = {Monday, Tuesday, Wednesday, Thursday, Friday} and B = {Friday, Saturday, Sunday}, find the indicated set.

11. $A \cap B$ **12.** $A \cup B$
13. B' **14.** A'
15. $A' \cup B$ **16.** $A \cap B'$

In Exercises 17–26, use a Venn diagram like the one in Figure 2.15 to shade in the region corresponding to the indicated set.

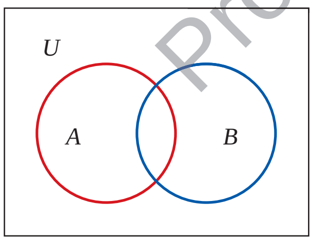


FIGURE 2.15 Two overlapping circles.

17. $A \cap B$ **18.** $A \cup B$
19. A' **20.** B'
21. $A \cup B'$ **22.** $A' \cup B$
23. $A' \cap B$ **24.** $A \cap B'$
25. $A' \cup B'$ **26.** $A' \cap B'$

27. Suppose $n(U) = 150$, $n(A) = 37$, and $n(B) = 84$.

a. If $n(A \cup B) = 100$, find $n(A \cap B)$ and draw a Venn diagram illustrating the composition of U.

b. If $n(A \cup B) = 121$, find $n(A \cap B)$ and draw a Venn diagram illustrating the composition of U.

28. Suppose $n(U) = w$, $n(A) = x$, $n(B) = y$, and $n(A \cup B) = z$.

a. Why must x be less than or equal to z?

b. If $A \neq U$ and $B \neq U$, fill in the blank with the most appropriate symbol: $<$, $>$, $\leq$, or $\geq$.

w_____z, w_____y, y_____z, x_____w

c. Find $n(A \cap B)$ and draw a Venn diagram illustrating the composition of U.

29. In a recent transportation survey, 500 high school seniors were asked to check the appropriate box or boxes on the following form:

☐ I own an automobile.
☐ I own a motorcycle.

The results were tabulated as follows: 102 students checked the automobile box, 147 checked the motorcycle box, and 21 checked both boxes.

a. Draw a Venn diagram illustrating the results of the survey.

b. What percent of these students own an automobile or a motorcycle?

30. In a recent market research survey, 500 married couples were asked to check the appropriate box or boxes on the following form:

☐ We own a DVD player.
☐ We own a microwave oven.

The results were tabulated as follows: 301 couples checked the DVD player box, 394 checked the microwave oven box, and 217 checked both boxes.

a. Draw a Venn diagram illustrating the results of the survey.

b. What percent of these couples own a DVD player or a microwave oven?

31. In a recent socioeconomic survey, 700 married women were asked to check the appropriate box or boxes on the following form:

☐ I have a career.
☐ I have a child.

The results were tabulated as follows: 285 women checked the child box, 316 checked the career box, and 196 were blank (no boxes were checked).

a. Draw a Venn diagram illustrating the results of the survey.

b. What percent of these women had both a child and a career?

32. In a recent health survey, 700 single men in their twenties were asked to check the appropriate box or boxes on the following form:

> ☐ I am a member of a private gym.
> ☐ I am a vegetarian.

The results were tabulated as follows: 349 men checked the gym box, 101 checked the vegetarian box, and 312 were blank (no boxes were checked).

a. Draw a Venn diagram illustrating the results of the survey.

b. What percent of these men were both members of a private gym and vegetarians?

For Exercises 33–36, let

$U = \{x \mid x$ is the name of one of the states in the United States$\}$

$A = \{x \mid x \in U$ and x begins with the letter A$\}$

$I = \{x \mid x \in U$ and x begins with the letter I$\}$

$M = \{x \mid x \in U$ and x begins with the letter M$\}$

$N = \{x \mid x \in U$ and x begins with the letter N$\}$

$O = \{x \mid x \in U$ and x begins with the letter O$\}$

33. Find $n(M')$.

34. Find $n(A \cup N)$.

35. Find $n(I' \cap O')$.

36. Find $n(M \cap I)$.

For Exercises 37–40, let

$U = \{x \mid x$ is the name of one of the months in a year$\}$

$J = \{x \mid x \in U$ and x begins with the letter J$\}$

$Y = \{x \mid x \in U$ and x ends with the letter Y$\}$

$V = \{x \mid x \in U$ and x begins with a vowel$\}$

$R = \{x \mid x \in U$ and x ends with the letter R$\}$

37. Find $n(R')$.

38. Find $n(J \cap V)$.

39. Find $n(J \cup Y)$.

40. Find $n(V \cap R)$.

In Exercises 41–50, determine how many cards, in an ordinary deck of fifty-two, fit the description. (If you are unfamiliar with playing cards, see the end of Section 3.1 for a description of a standard deck.)

41. spades or aces

42. clubs or 2's

43. face cards or black

44. face cards or diamonds

45. face cards and black

46. face cards and diamonds

47. aces or 8's

48. 3's or 6's

49. aces and 8's

50. 3's and 6's

51. Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.

a. Find $A \cap B$.

b. Find $A \cup B$.

c. In general, if $E \cap F = E$, what must be true concerning sets E and F?

d. In general, if $E \cup F = F$, what must be true concerning sets E and F?

52. Fill in the blank, and give an example to support your answer.

a. If $A \subset B$, then $A \cap B =$ _____.

b. If $A \subset B$, then $A \cup B =$ _____.

53. **a.** List all subsets of $A = \{a\}$. How many subsets does A have?

b. List all subsets of $A = \{a, b\}$. How many subsets does A have?

c. List all subsets of $A = \{a, b, c\}$. How many subsets does A have?

d. List all subsets of $A = \{a, b, c, d\}$. How many subsets does A have?

e. Is there a relationship between the cardinal number of set A and the number of subsets of set A?

f. How many subsets does $A = \{a, b, c, d, e, f\}$ have?

HINT: Use your answer to part (e).

54. Prove the Cardinal Number Formula for the Complement of a Set.

HINT: Apply the Union/Intersection Formula to A and A'.

Answer the following questions using complete sentences and your own words.

• CONCEPT QUESTIONS

55. If $A \cap B = \emptyset$, what is the relationship between sets A and B?

56. If $A \cup B = \emptyset$, what is the relationship between sets A and B?

57. Explain the difference between $\{0\}$ and $\emptyset$.

58. Explain the difference between 0 and $\{0\}$.

59. Is it possible to have $A \cap A = \emptyset$?

60. What is the difference between proper and improper subsets?

61. A set can be described by two methods: the roster method and set-builder notation. When is it advantageous to use the roster method? When is it advantageous to use set-builder notation?

62. Translate the following symbolic expressions into English sentences.

a. $x \in (A \cap B) \leftrightarrow (x \in A \wedge x \in B)$

b. $(x \in A') \leftrightarrow \sim (x \in A)$

c. $(A \subseteq B) \leftrightarrow (x \in A \rightarrow x \in B)$

• HISTORY QUESTIONS

63. In what academic field was John Venn a professor? Where did he teach?

64. What was one of John Venn's main contributions to the field of logic? What new benefits did it offer?

THE NEXT LEVEL

If a person wants to pursue an advanced degree (something beyond a bachelor's or four-year degree), chances are the person must take a standardized exam to gain admission to a school or to be admitted into a specific program. These exams are intended to measure verbal, Quantitative, and analytical skills that have developed throughout a person's life. Many classes and study guides are available to help people prepare for the exams. The following questions are typical of those found in the study guides.

Exercises 65–69 refer to the following: Two collectors, John and Juneko, are each selecting a group of three posters from a group of seven movie posters: J, K, L, M, N, O, *and* P. *No poster can be in both groups. The selections made by John and Juneko are subject to the following restrictions:*

- If K is in John's group, M must be in Juneko's group.
- If N is in John's group, P must be in Juneko's group.
- J and P cannot be in the same group.
- M and O cannot be in the same group.

65. Which of the following pairs of groups selected by John and Juneko conform to the restrictions?

	John	**Juneko**
a.	J, K, L	M, N, O
b.	J, K, P	L, M, N
c.	K, N, P	J, M, O
d.	L, M, N	K, O, P
e.	M, O, P	J, K, N

66. If N is in John's group, which of the following could not be in Juneko's group?

a. J **b.** K **c.** L **d.** M **e.** P

67. If K and N are in John's group, Juneko's group must consist of which of the following?

a. J, M, and O

b. J, O, and P

c. L, M, and P

d. L, O, and P

e. M, O, and P

68. If J is in Juneko's group, which of the following is true?

a. K cannot be in John's group.

b. N cannot be in John's group.

c. O cannot be in Juneko's group.

d. P must be in John's group.

e. P must be in Juneko's group.

69. If K is in John's group, which of the following is true?

a. J must be in John's group.

b. O must be in John's group.

c. L must be in Juneko's group.

d. N cannot be in John's group.

e. O cannot be in Juneko's group.

2.2 Applications of Venn Diagrams

OBJECTIVES

- Use Venn diagrams to analyze the results of surveys
- Develop and apply De Morgan's Laws of complements

As we have seen, Venn diagrams are very useful tools for visualizing the relationships between sets. They can be used to establish general formulas involving set operations and to determine the cardinal numbers of sets. Venn diagrams are particularly useful in survey analysis.

Surveys

Surveys are often used to divide people or objects into categories. Because the categories sometimes overlap, people can fall into more than one category. Venn diagrams and the formulas for cardinal numbers can help researchers organize the data.

EXAMPLE 1

ANALYZING THE RESULTS OF A SURVEY: TWO SETS Has the advent of the DVD affected attendance at movie theaters? To study this question, Professor Redrum's film class conducted a survey of people's movie-watching habits. He had his students ask hundreds of people between the ages of sixteen and forty-five to check the appropriate box or boxes on the following form:

☐ I watched a movie in a theater during the past month.
☐ I watched a movie on a DVD during the past month.

After the professor had collected the forms and tabulated the results, he told the class that 388 people had checked the theater box, 495 had checked the DVD box, 281 had checked both boxes, and 98 of the forms were blank. Giving the class only this information, Professor Redrum posed the following three questions.

a. What percent of the people surveyed watched a movie in a theater or on a DVD during the past month?
b. What percent of the people surveyed watched a movie in a theater only?
c. What percent of the people surveyed watched a movie on a DVD only?

SOLUTION

a. To calculate the desired percentages, we must determine $n(U)$, the total number of people surveyed. This can be accomplished by drawing a Venn diagram. Because the survey divides people into two categories (those who watched a movie in a theater and those who watched a movie on a DVD), we need to define two sets. Let

$$T = \{\text{people} \mid \text{the person watched a movie in a theater}\}$$
$$D = \{\text{people} \mid \text{the person watched a movie on a DVD}\}$$

Now translate the given survey information into the symbols for the sets and attach their given cardinal numbers: $n(T) = 388$, $n(D) = 495$, and $n(T \cap D) = 281$.

Our first goal is to find $n(U)$. To do so, we will fill in the cardinal numbers of all regions of a Venn diagram consisting of two overlapping circles (because we are dealing with two sets). The intersection of T and D consists of 281 people, so we draw two overlapping circles and fill in 281 as the number of elements in common (see Figure 2.16).

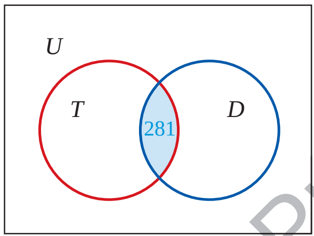


FIGURE 2.16
$n(T \cap D) = 281$.

Because we were given $n(T) = 388$ and know that $n(T \cap D) = 281$, the difference $388 - 281 = 107$ tells us that 107 people watched a movie in a theater but did not watch a movie on a DVD. We fill in 107 as the number of people who watched a movie only in a theater (see Figure 2.17).

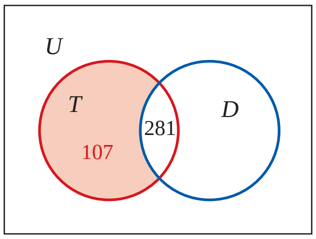


FIGURE 2.17
$n(T \cup D') = 107$.

Because $n(D) = 495$, the difference $495 - 281 = 214$ tells us that 214 people watched a movie on a DVD but not in a theater. We fill in 214 as the number of people who watched a movie only on a DVD (see Figure 2.18).

The only region remaining to be filled in is the region outside both circles. This region represents people who didn't watch a movie in a theater or on a DVD and is symbolized by $(T \cup D)'$. Because 98 people didn't check either box on the form, $n[(T \cup D)'] = 98$ (see Figure 2.19).

After we have filled in the Venn diagram with all the cardinal numbers, we readily see that $n(U) = 98 + 107 + 281 + 214 = 700$. Therefore, 700 people were in the survey.

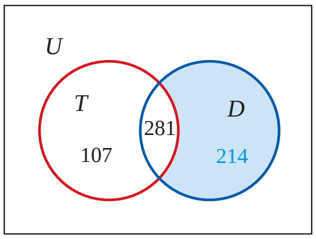


FIGURE 2.18
$n(D \cap T') = 214$.

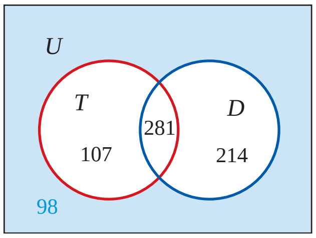


FIGURE 2.19
Completed Venn diagram.

To determine what *percent* of the people surveyed watched a movie in a theater *or* on a DVD during the past month, simply divide $n(T \cup D)$ by $n(U)$:

$$\frac{n(T \cup D)}{n(U)} = \frac{107 + 281 + 214}{700} = \frac{602}{700} = 0.86$$

Therefore, exactly 86% of the people surveyed watched a movie in a theater or on a DVD during the past month.

b. To find what *percent* of the people surveyed watched a movie in a theater only, divide 107 (the number of people who watched a movie in a theater only) by $n(U)$:

$$\frac{107}{700} = 0.152857142\ldots$$
$$\approx 0.153 \text{ (rounding off to three decimal places)}$$

Approximately 15.3% of the people surveyed watched a movie in a theater only.

c. Because 214 people watched a movie on DVD only, $214/700 = 0.305714285\ldots$, or approximately 30.6%, of the people surveyed watched a movie on DVD only.

When you solve a cardinal number problem (a problem that asks, "How many?" or "What percent?") involving a universal set that is divided into various categories (for instance, a survey), use the following general steps.

SOLVING A CARDINAL NUMBER PROBLEM

A cardinal number problem is a problem in which you are asked, "How many?" or "What percent?"

1. Define a set for each category in the universal set. If a category and its negation are both mentioned, define one set A and utilize its complement A'.
2. Draw a Venn diagram with as many overlapping circles as the number of sets you have defined.
3. Write down all the given cardinal numbers corresponding to the various given sets.
4. Starting with the innermost overlap, fill in each region of the Venn diagram with its cardinal number.
5. In answering a "what percent" problem, round off your answer to the nearest tenth of a percent.

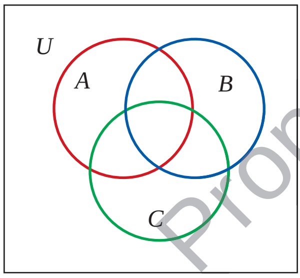


FIGURE 2.20
Three overlapping circles.

When we are working with three sets, we must account for all possible intersections of the sets. Hence, in such cases, we will use the Venn diagram shown in Figure 2.20

EXAMPLE 2

ANALYZING THE RESULTS OF A SURVEY: THREE SETS A consumer survey was conducted to examine patterns in ownership of notebook computers, cellular telephones, and DVD players. The following data were obtained: 213 people had notebook computers, 294 had cell phones, 337 had DVD players, 109 had all three, 64 had none, 198 had cell phones and DVD players, 382 had cell phones or notebook computers, and 61 had notebook computers and DVD players but no cell phones.

a. What percent of the people surveyed owned a notebook computer but no DVD player or cell phone?

b. What percent of the people surveyed owned a DVD player but no notebook computer or cell phone?

SOLUTION

a. To calculate the desired percentages, we must determine $n(U)$, the total number of people surveyed. This can be accomplished by drawing a Venn diagram. Because the survey divides people into three categories (those who own a notebook computer, those who own a cell phone, and those who own a DVD player), we need to define three sets. Let

$C = \{\text{people} \mid \text{the person owns a notebook computer}\}$

$T = \{\text{people} \mid \text{the person owns a cellular telephone}\}$

$D = \{\text{people} \mid \text{the person owns a DVD player}\}$

Now translate the given survey information into the symbols for the sets and attach their given cardinal numbers:

213 people had notebook computers →	$n(C) = 213$
294 had cellular telephones →	$n(T) = 294$
337 had DVD players →	$n(D) = 337$
109 had all three → (*C and T and D*)	$n(C \cap T \cap D) = 109$
64 had none → (not *C and* not *T and* not *D*)	$n(C' \cap T' \cap D') = 64$
198 had cell phones and DVD players → (*T and D*)	$n(T \cap D) = 198$
382 had cell phones or notebook computers → (*T or C*)	$n(T \cup C) = 382$
61 had notebook computers and DVD players but no cell phones → (*C and D and* not *T*)	$n(C \cap D \cap T') = 61$

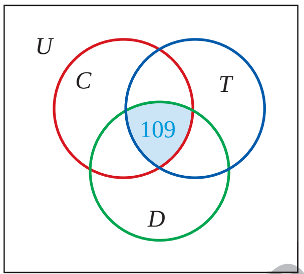


FIGURE 2.21
$n(C \cap T \cap D) = 109.$

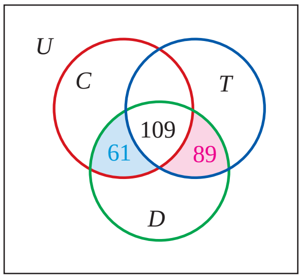


FIGURE 2.22
Determining cardinal numbers in a Venn diagram.

Our first goal is to find $n(U)$. To do so, we will fill in the cardinal numbers of all regions of a Venn diagram like that in Figure 2.20. We start by using information concerning membership in all three sets. Because the intersection of all three sets consists of 109 people, we fill in 109 in the region common to C and T and D (see Figure 2.21).

Next, we utilize any information concerning membership in two of the three sets. Because $n(T \cap D) = 198$, a total of 198 people are common to both T and D; some are in C, and some are not in C. Of these 198 people, 109 are in C (see Figure 2.21). Therefore, the difference $198 - 109 = 89$ gives the number not in C. Eighty-nine people are in T and D and *not* in C; that is, $n(T \cap D \cap C') = 89$. Concerning membership in the two sets C and D, we are given $n(C \cap D \cap T') = 61$. Therefore, we know that 61 people are in C and D and not in T (see Figure 2.22).

We are given $n(T \cup C) = 382$. From this number, we can calculate $n(T \cap C)$ by using the Union/Intersection Formula:

$$n(T \cup C) = n(T) + n(C) - n(T \cap C)$$
$$382 = 294 + 213 - n(T \cap C)$$
$$n(T \cap C) = 125$$

Therefore, a total of 125 people are in T and C; some are in D, and some are not in D. Of these 125 people, 109 are in D (see Figure 2.21). Therefore, the difference

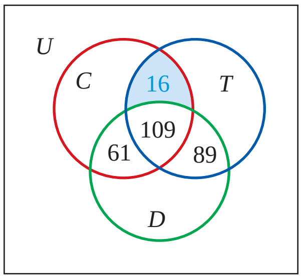


FIGURE 2.23 Determining cardinal numbers in a Venn diagram.

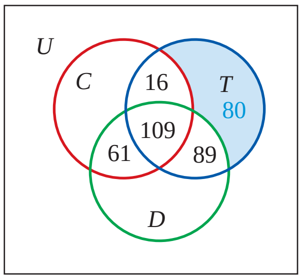


FIGURE 2.24 Determining cardinal numbers in a Venn diagram.

$125 - 109 = 16$ gives the number not in D. Sixteen people are in T and C and *not* in D; that is, $n(C \cap T \cap D') = 16$ (see Figure 2.23).

Knowing that a total of 294 people are in T (given $n(T) = 294$), we are now able to fill in the last region of T. The missing region (people in T only) has $294 - 109 - 89 - 16 = 80$ members; $n(T \cap C' \cap D') = 80$ (see Figure 2.24).

In a similar manner, we subtract the known pieces of C from $n(C) = 213$, which is given, and obtain $213 - 61 - 109 - 16 = 27$; therefore, 27 people are in C only. Likewise, to find the last region of D, we use $n(D) = 337$ (given) and obtain $337 - 89 - 109 - 61 = 78$; therefore, 78 people are in D only. Finally, the 64 people who own none of the items are placed "outside" the three circles (see Figure 2.25).

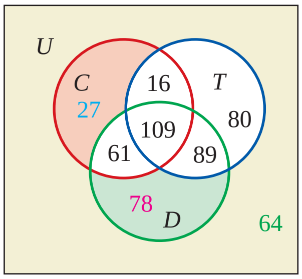


FIGURE 2.25
A completed Venn diagram.

By adding up the cardinal numbers of all the regions in Figure 2.25, we find that the total number of people in the survey is 524; that is, $n(U) = 524$.

Now, to determine what *percent* of the people surveyed owned only a notebook computer (no DVD player and no cell phone), we simply divide $n(C \cap D' \cap T')$ by $n(U)$:

$$\frac{n(C \cap D' \cap T')}{n(U)} = \frac{27}{524}$$
$$= 0.051526717\ldots$$

Approximately 5.2% of the people surveyed owned a notebook computer and did not own a DVD player or a cellular telephone.

b. To determine what *percent* of the people surveyed owned only a DVD player (no notebook computer and no cell phone), we divide $n(D \cap C' \cap T')$ by $n(U)$:

$$\frac{n(D \cap C' \cap T')}{n(U)} = \frac{78}{524}$$
$$= 0.148854961\ldots$$

Approximately 14.9% of the people surveyed owned a DVD player and did not own a notebook computer or a cell phone.

De Morgan's Laws

One of the basic properties of algebra is the distributive property:

$$a(b + c) = ab + ac$$

Given $a(b + c)$, the operation outside the parentheses can be distributed over the operation inside the parentheses. It makes no difference whether you add b and c first and then multiply the sum by a or first multiply each pair, a and b, a and c, and then add their products; the same result is obtained. Is there a similar property for the complement, union, and intersection of sets?

EXAMPLE 3

INVESTIGATING THE COMPLEMENT OF A UNION Suppose $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, and $B = \{2, 3, 4\}$.

a. For the given sets, does $(A \cup B)' = A' \cup B'$?
b. For the given sets, does $(A \cup B)' = A' \cap B'$?

SOLUTION

a. To find $(A \cup B)'$, we must first find $A \cup B$:

$$\begin{aligned} A \cup B &= \{1, 2, 3\} \cup \{2, 3, 4\} \\ &= \{1, 2, 3, 4\} \end{aligned}$$

The complement of $A \cup B$ (relative to the given universal set U) is

$$(A \cup B)' = \{5\}$$

To find $A' \cup B'$, we must first find A' and B':

$$A' = \{4, 5\} \quad \text{and} \quad B' = \{1, 5\}$$

The union of A' and B' is

$$\begin{aligned} A' \cup B' &= \{4, 5\} \cup \{1, 5\} \\ &= \{1, 4, 5\} \end{aligned}$$

Now, $\{5\} \neq \{1, 4, 5\}$; therefore, $(A \cup B)' \neq A' \cup B'$.

b. We find $(A \cup B)'$ as in part (a): $(A \cup B)' = \{5\}$. Now,

$$\begin{aligned} A' \cap B' &= \{4, 5\} \cap \{1, 5\} \\ &= \{5\} \end{aligned}$$

For the given sets, $(A \cup B)' = A' \cap B'$.

Part (a) of Example 3 shows that the operation of complementation *cannot* be explicitly distributed over the operation of union; that is, $(A \cup B)' \neq A' \cup B'$. However, part (b) of the example implies that there *may* be some relationship between the complement, union, and intersection of sets. The fact that $(A \cup B)' = A' \cap B'$ *for the given sets* A and B does not mean that it is true *for all sets* A and B. We will use a general Venn diagram to examine the validity of the statement $(A \cup B)' = A' \cap B'$.

When we draw two overlapping circles within a universal set, four regions are formed. Every element of the universal set U is in exactly one of the following regions, as shown in Figure 2.26:

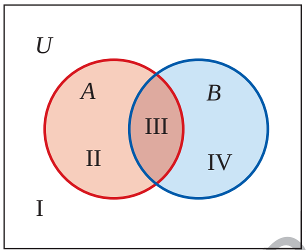


FIGURE 2.26
Four regions in a universal set U.

I in neither A nor B
II in A and not in B
III in both A and B
IV in B and not in A

The set $A \cup B$ consists of all elements in regions II, III, and IV. Therefore, the complement $(A \cup B)'$ consists of all elements in region I. A' consists of all elements in regions I and IV, and B' consists of the elements in regions I and II. Therefore, the elements common to both A' and B' are those in region I; that is, the set $A' \cap B'$ consists of all elements in region I. Since $(A \cup B)'$ and $A' \cap B'$ contain exactly the same elements (those in region I), the sets are equal; that is, $(A \cup B)' = A' \cap B'$ is true for all sets A and B.

The relationship $(A \cup B)' = A' \cap B'$ is known as one of **De Morgan's Laws.** Simply stated, "the complement of a union is the intersection of the complements." In a similar manner, it can be shown that $(A \cap B)' = A' \cup B'$ (see Exercise 33).

HISTORICAL NOTE

AUGUSTUS DE MORGAN, 1806–1871

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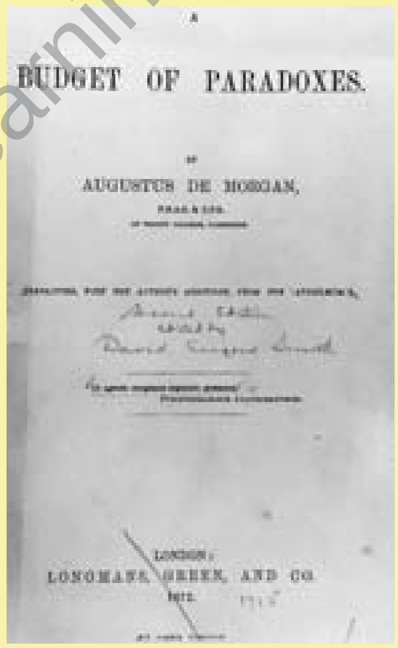
Being born blind in one eye did not stop Augustus De Morgan from becoming a well-read philosopher, historian, logician, and mathematician. De Morgan was born in Madras, India, where his father was working for the East India Company. On moving to England, De Morgan was educated at Cambridge, and at the age of twenty-two, he became the first professor of mathematics at the newly opened University of London (later renamed University College).

De Morgan viewed all of mathematics as an abstract study of symbols and of systems of operations applied to these symbols. While studying the ramifications of symbolic logic, De Morgan formulated the general properties of complementation that now bear his name. Not limited to symbolic logic, De Morgan's many works include books and papers on the foundations of algebra, differential calculus, and probability. He was known to be a jovial person who was fond of puzzles, and his witty and amusing book *A Budget of Paradoxes* still entertains readers today. Besides his accomplishments in the academic arena, De Morgan was an expert flutist, spoke five languages, and thoroughly enjoyed big-city life.

Knowing of his interest in probability, an actuary (someone who studies life expectancies and determines payments of premiums for insurance companies) once asked De Morgan a question concerning the probability that a certain group of people would be alive at a certain time. In his response, De Morgan employed a formula containing the number π. In amazement, the actuary responded, "That must surely be a delusion! What can a circle have to do with the number of people alive at a certain time?" De Morgan replied that π has numerous applications and occurrences in many diverse areas of mathematics. Because it was first defined and used in geometry, people are conditioned to accept the mysterious number only in reference to a circle. However, in the history of mathematics, if probability had been systematically studied before geometry and circles, our present-day interpretation of the number π would be entirely different. In addition to his accomplishments in logic and higher-level mathematics, De Morgan introduced a convention with which we are all familiar: In a paper written in 1845, he suggested the use of a slanted line to represent a fraction, such as 1/2 or 3/4.

De Morgan was a staunch defender of academic freedom and religious tolerance. While he was a student at Cambridge, his application for a fellowship was refused because he would not take and sign a theological oath. Later in life, he resigned his professorship as a protest against religious bias. (University College gave preferential treatment to members of the Church of England when textbooks were selected and did not have an open policy on religious philosophy.) Augustus De Morgan was a man who was unafraid to take a stand and make personal sacrifices when it came to principles he believed in.


A
BUDGET OF PARADOXES.
AUGUSTUS DE MORGAN,
LONDON:
LONGMANS, GREEN, AND CO.
1872.

Gematria is a mystic pseudoscience in which numbers are substituted for the letters in a name. De Morgan's book A Budget of Paradoxes *contains several gematria puzzles, such as, "Mr. Davis Thom found a young gentleman of the name of St. Claire busy at the Beast number: he forthwith added the letters in* σтκλαιρε *(the Greek spelling of St. Claire) and found 666." (Verify this by using the Greek numeral system.)*

DE MORGAN'S LAWS

For any sets A and B,

$$(A \cup B)' = A' \cap B'$$

That is, the complement of a union is the intersection of the complements. Also,

$$(A \cap B)' = A' \cup B'$$

That is, the complement of an intersection is the union of the complements.

TOPIC X

BLOOD TYPES: *SET THEORY IN THE REAL WORLD*

Human blood types are a classic example of set theory. As you may know, there are four categories (or sets) of blood types: A, B, AB, and O. Knowing someone's blood type is extremely important in case a blood transfusion is required; if blood of two different types is combined, the blood cells may begin to clump together, with potentially fatal consequences! (Do you know your blood type?)

What exactly are "blood types"? In the early 1900s, the Austrian scientist Karl Landsteiner observed the presence (or absence) of two distinct chemical molecules on the surface of all red blood cells in numerous samples of human blood. Consequently, he labeled one molecule "A" and the other "B." The presence or absence of these specific molecules is the basis of the universal classification of blood types. Specifically, blood samples containing only the A molecule are labeled type A, whereas those containing only the B molecule are labeled type B. If a blood sample contains both molecules (A and B) it is labeled type AB; and if neither is present, the blood is typed as O. The presence (or absence) of these molecules can be depicted in a standard Venn diagram as shown in Figure 2.27. In the notation of set operations, type A blood is denoted $A \cap B'$, type B is $B \cap A'$, type AB is $A \cap B$, and type O is $A' \cap B'$.

If a specific blood sample is mixed with blood containing a blood molecule (A or B) that it does not already have, the presence of the foreign molecule may cause the mixture of blood to clump. For example, type A blood cannot be mixed with any blood containing the B molecule (type B or type AB). Therefore, a person with type A blood can receive a transfusion only of type A or type O blood. Consequently, a person with type AB blood may receive a transfusion of any blood type; type AB is referred to as the "universal receiver." Because type O blood contains neither the A nor the B molecule, all blood types are compatible with type O blood; type O is referred to as the "universal donor."

It is not uncommon for scientists to study rhesus monkeys in an effort to learn more about human physiology. In so doing, a certain blood protein was discovered in rhesus monkeys. Subsequently, scientists found that the blood of some people contained this protein, whereas the blood of others did not. The presence, or absence, of this protein in human blood is referred to as the *Rh factor;* blood containing the protein is labeled "Rh+", whereas "Rh–" indicates the absence of the protein. The Rh factor of human blood is especially important for expectant mothers; a fetus can develop problems if its parents have opposite Rh factors.

When a person's blood is typed, the designation includes both the regular blood type and the Rh factor. For instance, type AB– indicates the presence of both the A and B molecules (type AB), along with the absence of the rhesus protein; type O+ indicates the absence of both the A and B molecules (type O), along with the presence of the rhesus protein. Utilizing the Rh factor, there are eight possible blood types as shown in Figure 2.28.

We will investigate the occurrence and compatibility of the various blood types in Example 5 and in Exercises 35–43.

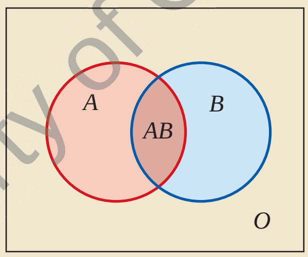


FIGURE 2.27 Blood types and the presence of the A and B molecules.

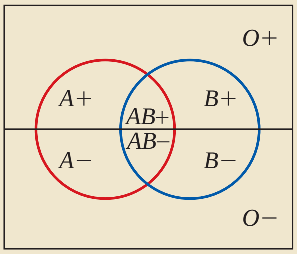


FIGURE 2.28 Blood types combined with the Rh factor.

EXAMPLE 4

APPLYING DE MORGAN'S LAW Suppose $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 3, 7, 8\}$, and $B = \{0, 4, 5, 7, 8, 9\}$. Use De Morgan's Law to find $(A' \cup B)'$.

SOLUTION

The complement of a union is equal to the intersection of the complements; therefore, we have

$$\begin{aligned} (A' \cup B)' &= (A')' \cap B' && \textbf{De Morgan's Law} \\ &= A \cap B' && \mathbf{(A')' = A} \\ &= \{2, 3, 7, 8\} \cap \{1, 2, 3, 6\} \\ &= \{2, 3\} \end{aligned}$$

Notice that this problem could be done without using De Morgan's Law, but solving it would then involve finding first A', then $A' \cup B$, and finally $(A' \cup B)'$. This method would involve more work. (Try it!)

EXAMPLE 5

INVESTIGATING BLOOD TYPES IN THE UNITED STATES The American Red Cross has compiled a massive database of the occurrence of blood types in the United States. Their data indicate that on average, out of every 100 people in the United States, 44 have the A molecule, 15 have the B molecule, and 45 have neither the A nor the B molecule. What percent of the U.S. population have the following blood types?

a. Type O? **b.** Type AB? **c.** Type A? **d.** Type B?

SOLUTION

a. First, we define the appropriate sets. Let

$A = \{\text{Americans} \mid \text{the person has the A molecule}\}$

$B = \{\text{Americans} \mid \text{the person has the B molecule}\}$

We are given the following cardinal numbers: $n(U) = 100$, $n(A) = 44$, $n(B) = 15$, and $n(A' \cap B') = 45$. Referring to Figure 2.27, and given that 45 people (out of 100) have neither the A molecule nor the B molecule, we conclude that 45 of 100 people, or 45%, have type O blood as shown in Figure 2.29.

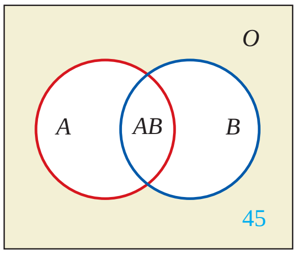


FIGURE 2.29 Forty-five of 100 (45%) have type O blood.

b. Applying De Morgan's Law to the Complement Formula, we have the following.

$$n(A \cup B) + n[(A \cup B)'] = n(U) \quad \text{Complement Formula}$$
$$n(A \cup B) + n(A' \cap B') = n(U) \quad \text{applying De Morgan's Law}$$
$$n(A \cup B) + 45 = 100 \quad \text{substituting known values}$$

Therefore, $n(A \cup B) = 55$.

Now, use the Union/Intersection Formula.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \quad \text{Union/Intersection Formula}$$
$$55 = 44 + 15 - n(A \cap B) \quad \text{substituting known values}$$
$$n(A \cap B) = 44 + 15 - 55 \quad \text{adding } n(A \cap B) \text{ and subtracting 55}$$

Therefore, $n(A \cap B) = 4$. This means that 4 people (of 100) have both the A and the B molecules; that is, 4 of 100 people, or 4%, have type AB blood. See Figure 2.30.

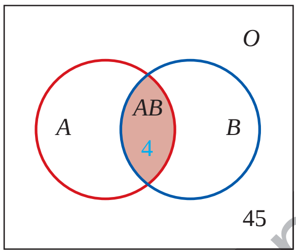


FIGURE 2.30 Four in 100 (4%) have type AB blood.

c. Knowing that a total of 44 people have the A molecule, that is, $n(A) = 44$, we subtract $n(A \cap B) = 4$ and conclude that 40 have *only* the A molecule.

Therefore, $n(A \cap B') = 40$. This means that 40 people (of 100) have only the A molecule; that is, 40 of 100 people, or 40%, have type A blood. See Figure 2.31.

d. Knowing that a total of 15 people have the B molecule, that is, $n(B) = 15$, we subtract $n(A \cap B) = 4$ and conclude that 11 have *only* the B molecule.

Therefore, $n(B \cap A') = 11$. This means that 11 people (of 100) have only the B molecule; that is, 11 of 100 people, or 11%, have type B blood (see Figure 2.32).

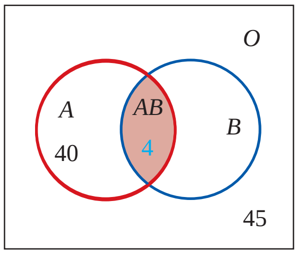


FIGURE 2.31 Forty in 100 (40%) have type A blood.

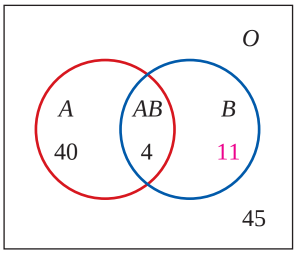


FIGURE 2.32 Eleven in 100 (11%) have type B blood.

The occurrence of blood types in the United States is summarized in Figure 2.33.

Blood Type	O	A	B	AB
Occurrence	45%	40%	11%	4%

FIGURE 2.33 Occurrence of blood types in the United States.

The occurrence of blood types given in Figure 2.33 can be further categorized by including the Rh factor. According to the American Red Cross, out of every 100 people in the United States, blood types and Rh factors occur at the rates shown in Figure 2.34.

Blood Type	O+	O–	A+	A–	B+	B–	AB+	AB–
Occurrence	38	7	34	6	9	2	3	1

FIGURE 2.34 Blood types per 100 people in the United States. (*Source:* American Red Cross.)

2.2 Exercises

1. A survey of 200 people yielded the following information: 94 people owned a DVD player, 127 owned a microwave oven, and 78 owned both. How many people owned the following?

a. a DVD player or a microwave oven

b. a DVD player but not a microwave oven

c. a microwave oven but not a DVD player

d. neither a DVD player nor a microwave oven

2. A survey of 300 workers yielded the following information: 231 workers belonged to a union, and 195 were Democrats. If 172 of the union members were Democrats, how many workers were in the following situations?

a. belonged to a union or were Democrats

b. belonged to a union but were not Democrats

c. were Democrats but did not belong to a union

d. neither belonged to a union nor were Democrats

3. The records of 1,492 high school graduates were examined, and the following information was obtained: 1,072 graduates took biology, and 679 took geometry. If 271 of those who took geometry did not take biology, how many graduates took the following?

a. both classes

b. at least one of the classes

c. biology but not geometry

d. neither class

4. A department store surveyed 428 shoppers, and the following information was obtained: 214 shoppers made a purchase, and 299 were satisfied with the service they received. If 52 of those who made a purchase were not satisfied with the service, how many shoppers did the following?

a. made a purchase and were satisfied with the service

b. made a purchase or were satisfied with the service

c. were satisfied with the service but did not make a purchase

d. were not satisfied and did not make a purchase

5. In a survey, 674 adults were asked what television programs they had recently watched. The following information was obtained: 226 adults watched neither the Big Game nor the New Movie, and 289 watched the New Movie. If 183 of those who watched the New Movie did not watch the Big Game, how many of the surveyed adults watched the following?

a. both programs

b. at least one program

c. the Big Game

d. the Big Game but not the New Movie

6. A survey asked 816 college freshmen whether they had been to a movie or eaten in a restaurant during the past week. The following information was obtained: 387 freshmen had been to neither a movie nor a restaurant, and 266 had been to a movie. If 92 of those who had been to a movie had not been to a restaurant, how many of the surveyed freshmen had been to the following?

a. both a movie and a restaurant

b. a movie or a restaurant

c. a restaurant

d. a restaurant but not a movie

7. A local 4-H club surveyed its members, and the following information was obtained: 13 members had rabbits, 10 had goats, 4 had both rabbits and goats, and 18 had neither rabbits nor goats.

a. What percent of the club members had rabbits or goats?

b. What percent of the club members had only rabbits?

c. What percent of the club members had only goats?

8. A local anime fan club surveyed its members regarding their viewing habits last weekend, and the following information was obtained: 30 members had watched an episode of *Naruto*, 44 had watched an episode of *Death Note*, 21 had watched both an episode of *Naruto* and an episode of *Death Note*, and 14 had watched neither *Naruto* nor *Death Note*.

a. What percent of the club members had watched *Naruto* or *Death Note?*

b. What percent of the club members had watched only *Naruto?*

c. What percent of the club members had watched only *Death Note?*

9. A recent survey of w shoppers (that is, $n(U) = w$) yielded the following information: x shoppers shopped at Sears, y shopped at JCPenney's, and z shopped at both. How many people shopped at the following?

a. Sears or JCPenney's

b. only Sears

c. only JCPenney's

d. neither Sears nor JCPenney's

10. A recent transportation survey of w urban commuters (that is, $n(U) = w$) yielded the following information: x commuters rode neither trains nor buses, y rode trains, and z rode only trains. How many people rode the following?

a. trains and buses

b. only buses

c. buses

d. trains or buses

11. A consumer survey was conducted to examine patterns in ownership of laptop computers, cellular telephones, and DVD players. The following data were obtained: 313 people had laptop computers, 232 had cell phones, 269 had DVD players, 69 had all three, 64 had none, 98 had cell phones and DVD players, 57 had cell phones but no computers or DVD players, and 104 had computers and DVD players but no cell phones.

a. What percent of the people surveyed owned a cell phone?

b. What percent of the people surveyed owned only a cell phone?

12. In a recent survey of monetary donations made by college graduates, the following information was obtained: 95 graduates had donated to a political campaign, 76 had donated to assist medical research, 133 had donated to help preserve the environment, 25 had donated to all three, 22 had donated to none of the three, 38 had donated to a political campaign and to medical research, 46 had donated to medical research and to preserve the environment, and 54 had donated to a political campaign and to preserve the environment.

a. What percent of the college graduates donated to none of the three listed causes?

b. What percent of the college graduates donated to exactly one of the three listed causes?

13. Recently, Green Day, the Kings of Leon, and the Black Eyed Peas had concert tours in the United States. A large group of college students was surveyed, and the following information was obtained: 381 students saw Black Eyed Peas, 624 saw the Kings of Leon, 712 saw Green Day, 111 saw all three, 513 saw none, 240 saw only Green Day, 377 saw Green Day and the Kings of Leon, and 117 saw the Kings of Leon and Black Eyed Peas but not Green Day.

a. What percent of the college students saw at least one of the bands?

b. What percent of the college students saw exactly one of the bands?

14. Dr. Hawk works in an allergy clinic, and his patients have the following allergies: 68 patients are allergic to dairy products, 93 are allergic to pollen, 91 are allergic to animal dander, 31 are allergic to all three, 29 are allergic only to pollen, 12 are allergic only to dairy products, and 40 are allergic to dairy products and pollen.

a. What percent of Dr. Hawk's patients are allergic to animal dander?

b. What percent of Dr. Hawk's patients are allergic only to animal dander?

15. When the members of the Eye and I Photo Club discussed what type of film they had used during the past month, the following information was obtained: 77 members used black and white, 24 used only black and white, 65 used color, 18 used only color, 101 used black and white or color, 27 used infrared, 9 used all three types, and 8 didn't use any film during the past month.

a. What percent of the members used only infrared film?

b. What percent of the members used at least two of the types of film?

16. After leaving the polls, many people are asked how they voted. (This is called an *exit poll.*) Concerning Propositions A, B, and C, the following information was obtained: 294 people voted yes on A, 90 voted yes only on A, 346 voted yes on B, 166 voted yes only on B, 517 voted yes on A or B, 339 voted yes on C, no one voted yes on all three, and 72 voted no on all three.

a. What percent of the voters in the exit poll voted no on A?

b. What percent of the voters voted yes on more than one proposition?

17. In a recent survey, consumers were asked where they did their gift shopping. The following results were obtained: 621 consumers shopped at Macy's, 513 shopped at Emporium, 367 shopped at Nordstrom, 723 shopped at Emporium or Nordstrom, 749 shopped at Macy's or Nordstrom, 776 shopped at Macy's or Emporium, 157 shopped at all three, 96 shopped at neither Macy's nor Emporium nor Nordstrom.

a. What percent of the consumers shopped at more than one store?

b. What percent of the consumers shopped exclusively at Nordstrom?

18. A company that specializes in language tutoring lists the following information concerning its English-speaking employees: 23 employees speak German; 25 speak French; 31 speak Spanish; 43 speak Spanish or French; 38 speak French or German; 46 speak German or Spanish; 8 speak Spanish, French, and German; and 7 speak English only.

a. What percent of the employees speak at least one language other than English?

b. What percent of the employees speak at least two languages other than English?

19. In a recent survey, people were asked which radio station they listened to on a regular basis. The following results were obtained: 140 people listened to WOLD (oldies), 95 listened to WJZZ (jazz), 134 listened to WTLK (talk show news), 235 listened to WOLD or WJZZ, 48 listened to WOLD and WTLK, 208 listened to WTLK or WJZZ, and 25 listened to none.

a. What percent of people in the survey listened only to WTLK on a regular basis?

b. What percent of people in the survey did not listen to WTLK on a regular basis?

20. In a recent health insurance survey, employees at a large corporation were asked, "Have you been a patient in a hospital during the past year, and if so, for what reason?" The following results were obtained: 494 employees had an injury, 774 had an illness, 1,254 had tests, 238 had an injury and an illness and tests, 700 had an illness and tests, 501 had tests and no injury or illness, 956 had an injury or illness, and 1,543 had not been a patient.

a. What percent of the employees had been patients in a hospital?

b. What percent of the employees had tests in a hospital?

In Exercises 21 and 22, use a Venn diagram like the one in Figure 2.35.

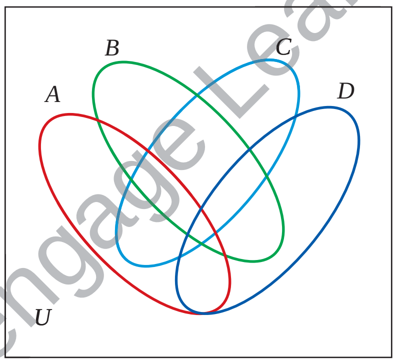


FIGURE 2.35 Four overlapping regions.

21. A survey of 136 pet owners yielded the following information: 49 pet owners own fish; 55 own a bird; 50 own a cat; 68 own a dog; 2 own all four; 11 own only fish; 14 own only a bird; 10 own fish and a bird; 21 own fish and a cat; 26 own a bird and a dog; 27 own a cat and a dog; 3 own fish, a bird, a cat, and no dog; 1 owns fish, a bird, a dog, and no cat; 9 own fish, a cat, a dog, and no bird; and 10 own a bird, a cat, a dog, and no fish. How many of the surveyed pet owners have no fish, no birds, no cats, and no dogs? (They own other types of pets.)

22. An exit poll of 300 voters yielded the following information regarding voting patterns on Propositions A, B, C, and D: 119 voters voted yes on A; 163 voted yes on B; 129 voted yes on C; 142 voted yes on D; 37 voted yes on all four; 15 voted yes on A only; 50 voted yes on B only; 59 voted yes on A and B; 70 voted yes on A and C; 82 voted yes on B and D; 93 voted yes on C and D; 10 voted yes on A, B, and C and no on D; 2 voted yes on A, B, and D and no on C; 16 voted yes on A, C, and D and no on B; and 30 voted yes on B, C, and D and no on A. How many of the surveyed voters voted no on all four propositions?

In Exercises 23–26, given the sets $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{0, 2, 4, 5, 9\}$, and $B = \{1, 2, 7, 8, 9\}$, use De Morgan's Laws to find the indicated sets.

23. $(A' \cup B)'$

24. $(A' \cap B)'$

25. $(A \cap B')'$

26. $(A \cup B')'$

In Exercises 27–32, use a Venn diagram like the one in Figure 2.36 to shade in the region corresponding to the indicated set.

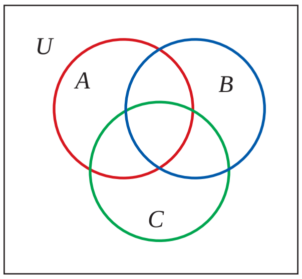


FIGURE 2.36 Three overlapping circles.

27. $A \cap B \cap C$

28. $A \cup B \cup C$

29. $(A \cup B)' \cap C$

30. $A \cap (B \cup C)'$

31. $B \cap (A \cup C')$

32. $(A' \cup B) \cap C'$

33. Using Venn diagrams, prove De Morgan's Law $(A \cap B)' = A' \cup B'$.

34. Using Venn diagrams, prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Use the data in Figure 2.34 to complete Exercises 35–39. Round off your answers to a tenth of a percent.

35. What percent of all people in the United States have blood that is

a. Rh positive? **b.** Rh negative?

36. Of all people in the United States who have type O blood, what percent are

a. Rh positive? **b.** Rh negative?

37. Of all people in the United States who have type A blood, what percent are

a. Rh positive? **b.** Rh negative?

38. Of all people in the United States who have type B blood, what percent are

a. Rh positive? **b.** Rh negative?

39. Of all people in the United States who have type AB blood, what percent are

a. Rh positive? **b.** Rh negative?

40. If a person has type A blood, what blood types may the person receive in a transfusion?

41. If a person has type B blood, what blood types may the person receive in a transfusion?

42. If a person has type AB blood, what blood types may the person receive in a transfusion?

43. If a person has type O blood, what blood types may the person receive in a transfusion?

Answer the following questions using complete sentences and your own words.

• HISTORY QUESTIONS

44. What notation did De Morgan introduce in regard to fractions?

45. Why did De Morgan resign his professorship at University College?

THE NEXT LEVEL

If a person wants to pursue an advanced degree (something beyond a bachelor's or four-year degree), chances are the person must take a standardized exam to gain admission to a school or to be admitted into a specific program. These exams are intended to measure verbal, quantitative, and analytical skills that have developed throughout a person's life. Many classes and study guides are available to help people prepare for the exams. The following questions are typical of those found in the study guides.

Exercises 46–52 refer to the following: A nonprofit organization's board of directors, composed of four women (Angela, Betty, Carmen, and Delores) and three men (Ed, Frank, and Grant), holds frequent meetings. A meeting can be held at Betty's house, at Delores's house, or at Frank's house.

- Delores cannot attend any meetings at Betty's house.
- Carmen cannot attend any meetings on Tuesday or on Friday.
- Angela cannot attend any meetings at Delores's house.
- Ed can attend only those meetings that Grant also attends.
- Frank can attend only those meetings that both Angela and Carmen attend.

46. If all members of the board are to attend a particular meeting, under which of the following circumstances can it be held?

a. Monday at Betty's

b. Tuesday at Frank's

c. Wednesday at Delores's

d. Thursday at Frank's

e. Friday at Betty's

47. Which of the following can be the group that attends a meeting on Wednesday at Betty's?
a. Angela, Betty, Carmen, Ed, and Frank
b. Angela, Betty, Ed, Frank, and Grant
c. Angela, Betty, Carmen, Delores, and Ed
d. Angela, Betty, Delores, Frank, and Grant
e. Angela, Betty, Carmen, Frank, and Grant

48. If Carmen and Angela attend a meeting but Grant is unable to attend, which of the following could be true?
a. The meeting is held on Tuesday.
b. The meeting is held on Friday.
c. The meeting is held at Delores's.
d. The meeting is held at Frank's.
e. The meeting is attended by six of the board members.

49. If the meeting is held on Tuesday at Betty's, which of the following pairs can be among the board members who attend?
a. Angela and Frank
b. Ed and Betty
c. Carmen and Ed
d. Frank and Delores
e. Carmen and Angela

50. If Frank attends a meeting on Thursday that is not held at his house, which of the following must be true?
a. The group can include, at most, two women.
b. The meeting is at Betty's house.
c. Ed is not at the meeting.
d. Grant is not at the meeting.
e. Delores is at the meeting.

51. If Grant is unable to attend a meeting on Tuesday at Delores's, what is the largest possible number of board members who can attend?
a. 1 **b.** 2 **c.** 3
d. 4 **e.** 5

52. If a meeting is held on Friday, which of the following board members *cannot* attend?
a. Grant **b.** Delores **c.** Ed
d. Betty **e.** Frank

WEB PROJECT

53. A person's Rh factor will limit the person's options regarding the blood types he or she may receive during a transfusion. Fill in the following chart. How does a person's Rh factor limit that person's options regarding compatible blood?

If Your Blood Type Is:	You Can Receive:
O+	
O–	
A+	
A–	
B+	
B–	
AB+	
AB–	

Some useful links for this web project are listed on the text web site:
www.cengage.com/math/johnson

2.3 Introduction to Combinatorics

OBJECTIVES

- Develop and apply the Fundamental Principle of Counting
- Develop and evaluate factorials

If you went on a shopping spree and bought two pairs of jeans, three shirts, and two pairs of shoes, how many new outfits (consisting of a new pair of jeans, a new shirt, and a new pair of shoes) would you have? A compact disc buyers' club sends you a brochure saying that you can pick any five CDs from a group of 50 of today's hottest sounds for only $1.99. How many different combinations can you choose? Six local bands have volunteered to perform at a benefit concert, and there is some concern over the order in which the bands will perform. How many different lineups are possible? The answers to questions like these can be obtained by listing all the possibilities or by using three shortcut counting methods: the **Fundamental Principle of Counting, combinations,** and **permutations**. Collectively, these methods are known as **combinatorics**. (Incidentally, the answers to the questions above are 12 outfits, 2,118,760 CD combinations, and 720 lineups.) In this section, we consider the first shortcut method.

The Fundamental Principle of Counting

Daily life requires that we make many decisions. For example, we must decide what food items to order from a menu, what items of clothing to put on in the morning, and what options to order when purchasing a new car. Often, we are asked to make a series of decisions: "Do you want soup or salad? What type of dressing? What type of vegetable? What entrée? What beverage? What dessert?" These individual components of a complete meal lead to the question "Given all the choices of soups, salads, dressings, vegetables, entrées, beverages, and desserts, what is the total number of possible dinner combinations?"

When making a series of decisions, how can you determine the total number of possible selections? One way is to list all the choices for each category and then match them up in all possible ways. To ensure that the choices are matched up in all possible ways, you can construct a **tree diagram.** A tree diagram consists of clusters of line segments, or *branches,* constructed as follows: A cluster of branches is drawn for each decision to be made such that the number of branches in each cluster equals the number of choices for the decision. For instance, if you must make two decisions and there are two choices for decision 1 and three choices for decision 2, the tree diagram would be similar to the one shown in Figure 2.37.

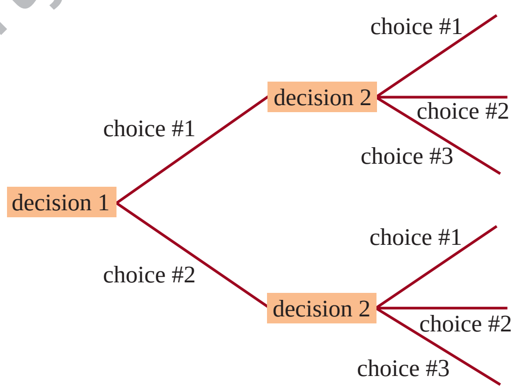


FIGURE 2.37 A tree diagram.

Although this method can be applied to all problems, it is very time consuming and impractical when you are dealing with a series of many decisions, each of which contains numerous choices. Instead of actually listing all possibilities via a tree diagram, using a shortcut method might be desirable. The following example gives a clue to finding such a shortcut.

EXAMPLE 1

DETERMINING THE TOTAL NUMBER OF POSSIBLE CHOICES IN A SERIES OF DECISIONS If you buy two pairs of jeans, three shirts, and two pairs of shoes, how many new outfits (consisting of a new pair of jeans, a new shirt, and a new pair of shoes) would you have?

SOLUTION

Because there are three categories, selecting an outfit requires a series of three decisions: You must select one pair of jeans, one shirt, and one pair of shoes. We will make our three decisions in the following order: jeans, shirt, and shoes. (The order in which the decisions are made does not affect the overall outfit.)

Our first decision (jeans) has two choices (jeans 1 or jeans 2); our tree starts with two branches, as in Figure 2.38.

Our second decision is to select a shirt, for which there are three choices. At each pair of jeans on the tree, we draw a cluster of three branches, one for each shirt, as in Figure 2.39.

Our third decision is to select a pair of shoes, for which there are two choices. At each shirt on the tree, we draw a cluster of two branches, one for each pair of shoes, as in Figure 2.40.

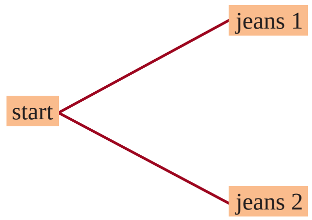


FIGURE 2.38 The first decision.

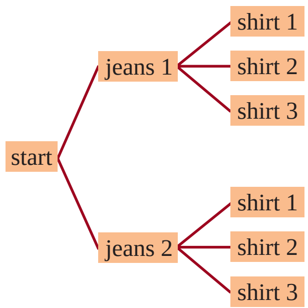


FIGURE 2.39 The second decision.

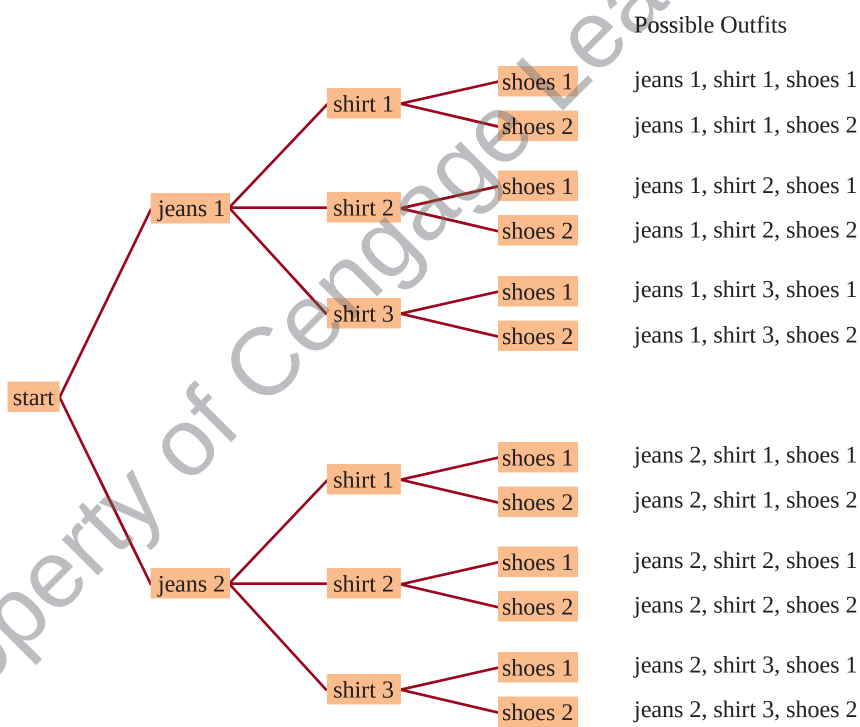


FIGURE 2.40 The third decision.

We have now listed all possible ways of putting together a new outfit; twelve outfits can be formed from two pairs of jeans, three shirts, and two pairs of shoes.

Referring to Example 1, note that each time a decision had to be made, the number of branches on the tree diagram was *multiplied* by a factor equal to the number of choices for the decision. Therefore, the total number of outfits could have been obtained by *multiplying* the number of choices for each decision:

$$\underbrace{2}_{\text{jeans}} \cdot \underbrace{3}_{\text{shirts}} \cdot \underbrace{2}_{\text{shoes}} = \underbrace{12}_{\text{outfits}}$$

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The generalization of this process of multiplication is called the Fundamental Principle of Counting.

THE FUNDAMENTAL PRINCIPLE OF COUNTING

The total number of possible outcomes of a series of decisions (making selections from various categories) is found by multiplying the number of choices for each decision (or category) as follows:

1. Draw a box for each decision.
2. Enter the number of choices for each decision in the appropriate box and multiply.

EXAMPLE 2

APPLYING THE FUNDAMENTAL PRINCIPLE OF COUNTING A serial number consists of two consonants followed by three nonzero digits followed by a vowel (A, E, I, O, U): for example, "ST423E" and "DD666E." Determine how many serial numbers are possible given the following conditions.

a. Letters and digits cannot be repeated in the same serial number.
b. Letters and digits can be repeated in the same serial number.

SOLUTION

a. Because the serial number has six symbols, we must make six decisions. Consequently, we must draw six boxes:

$$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$$

There are twenty-one different choices for the first consonant. Because the letters cannot be repeated, there are only twenty choices for the second consonant. Similarly, there are nine different choices for the first nonzero digit, eight choices for the second, and seven choices for the third. There are five different vowels, so the total number of possible serial numbers is

$$\underbrace{\boxed{21}\times\boxed{20}}_{\text{consonants}}\times\underbrace{\boxed{9}\times\boxed{8}\times\boxed{7}}_{\text{nonzero digits}}\times\underbrace{\boxed{5}}_{\text{vowel}} = 1{,}058{,}400$$

There are 1,058,400 possible serial numbers when the letters and digits cannot be repeated within a serial number.

b. Because letters and digits can be repeated, the number of choices does not decrease by one each time as in part (a). Therefore, the total number of possibilities is

$$\underbrace{\boxed{21}\times\boxed{21}}_{\text{consonants}}\times\underbrace{\boxed{9}\times\boxed{9}\times\boxed{9}}_{\text{nonzero digits}}\times\underbrace{\boxed{5}}_{\text{vowel}} = 1{,}607{,}445$$

There are 1,607,445 possible serial numbers when the letters and digits can be repeated within a serial number.

Factorials

EXAMPLE 3

APPLYING THE FUNDAMENTAL PRINCIPLE OF COUNTING Three students rent a three-bedroom house near campus. One of the bedrooms is very desirable (it has its own bath), one has a balcony, and one is undesirable (it is very small). In how many ways can the housemates choose the bedrooms?

SOLUTION Three decisions must be made: who gets the room with the bath, who gets the room with the balcony, and who gets the small room. Using the Fundamental Principle of Counting, we draw three boxes and enter the number of choices for each decision. There are three choices for who gets the room with the bath. Once that decision has been made, there are two choices for who gets the room with the balcony, and finally, there is only one choice for the small room.

$$\boxed{3} \times \boxed{2} \times \boxed{1} = 6$$

There are six different ways in which the three housemates can choose the three bedrooms.

Combinatorics often involve products of the type $3 \cdot 2 \cdot 1 = 6$, as seen in Example 3. This type of product is called a **factorial,** and the product $3 \cdot 2 \cdot 1$ is written as 3!. In this manner, $4! = 4 \cdot 3 \cdot 2 \cdot 1 \,(= 24)$, and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \,(= 120)$.

FACTORIALS

If n is a positive integer, then *n factorial,* denoted by $n!$, is the product of all positive integers less than or equal to n.

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$$

As a special case, we define $0! = 1$.

Many scientific calculators have a button that will calculate a factorial. Depending on your calculator, the button will look like $\boxed{x!}$ or $\boxed{n!}$, and you might have to press a $\boxed{\text{shift}}$ or $\boxed{\text{2nd}}$ button first. For example, to calculate 6!, type the number 6, press the factorial button, and obtain 720. To calculate a factorial on most graphing calculators, do the following:

- Type the value of n. (For example, type the number 6.)
- Press the $\boxed{\text{MATH}}$ button.
- Press the right arrow button $\boxed{\rightarrow}$ as many times as necessary to highlight $\boxed{\text{PRB}}$.
- Press the down arrow $\boxed{\downarrow}$ as many times as necessary to highlight the "!" symbol, and press $\boxed{\text{ENTER}}$.
- Press $\boxed{\text{ENTER}}$ to execute the calculation.

To calculate a factorial on a Casio graphing calculator, do the following:

- Press the $\boxed{\text{MENU}}$ button; this gives you access to the main menu.
- Press 1 to select the RUN mode; this mode is used to perform arithmetic operations.
- Type the value of n. (For example, type the number 6.)
- Press the $\boxed{\text{OPTN}}$ button; this gives you access to various options displayed at the bottom of the screen.
- Press the $\boxed{\text{F6}}$ button to see more options (i.e., $\boxed{\rightarrow}$).
- Press the $\boxed{\text{F3}}$ button to select probability options (i.e., $\boxed{\text{PROB}}$).
- Press the $\boxed{\text{F1}}$ button to select factorial (i.e., $\boxed{x!}$).
- Press the $\boxed{\text{EXE}}$ button to execute the calculation.

The factorial symbol "n!" was first introduced by Christian Kramp (1760–1826) of Strasbourg in his *Élements d'Arithmétique Universelle* (1808). Before the introduction of this "modern" symbol, factorials were commonly denoted by $\underline{|n}$. However, printing presses of the day had difficulty printing this symbol; consequently, the symbol n! came into prominence because it was relatively easy for a typesetter to use.

EXAMPLE 4

EVALUATING FACTORIALS Find the following values.

a. $6!$ **b.** $\frac{8!}{5!}$ **c.** $\frac{8!}{3! \cdot 5!}$

SOLUTION

a. $$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Therefore, $6! = 720$.

6 $\boxed{x!}$

6 $\boxed{\text{MATH}}$ $\boxed{\text{PRB}}$ $\boxed{!}$ $\boxed{\text{ENTER}}$

Casio 6 $\boxed{\text{OPTN}}$ $\boxed{\rightarrow}$ (i.e., $\boxed{\text{F6}}$) $\boxed{\text{PROB}}$ (i.e., $\boxed{\text{F3}}$) $\boxed{x!}$ (i.e., $\boxed{\text{F1}}$) $\boxed{\text{EXE}}$

b. $$\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$
$$= 8 \cdot 7 \cdot 6$$
$$= 336$$

Therefore, $\frac{8!}{5!} = 336$.

Using a calculator, we obtain the same result.

8 $\boxed{x!}$ $\boxed{\div}$ 5 $\boxed{x!}$ $\boxed{=}$

8 $\boxed{\text{MATH}}$ $\boxed{\text{PRB}}$ $\boxed{!}$ $\boxed{\div}$ 5 $\boxed{\text{MATH}}$ $\boxed{\text{PRB}}$ $\boxed{!}$ $\boxed{\text{ENTER}}$

c. $$\frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$$
$$= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(3 \cdot 2 \cdot 1)\,(\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1})}$$
$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$$
$$= 56$$

Therefore, $\frac{8!}{3! \cdot 5!} = 56$.

Using a calculator, we obtain the same result.

8 $\boxed{x!}$ $\boxed{\div}$ $\boxed{(}$ 3 $\boxed{x!}$ $\boxed{\times}$ 5 $\boxed{x!}$ $\boxed{)}$ $\boxed{=}$

8 $\boxed{\text{MATH}}$ $\boxed{\text{PRB}}$ $\boxed{!}$ $\boxed{\div}$ $\boxed{(}$ 3 $\boxed{\text{MATH}}$ $\boxed{\text{PRB}}$ $\boxed{!}$ $\boxed{\times}$ 5 $\boxed{\text{MATH}}$ $\boxed{\text{PRB}}$ $\boxed{!}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

2.3 Exercises

1. A nickel, a dime, and a quarter are tossed.
   a. Use the Fundamental Principle of Counting to determine how many different outcomes are possible.
   b. Construct a tree diagram to list all possible outcomes.
2. A die is rolled, and a coin is tossed.
   a. Use the Fundamental Principle of Counting to determine how many different outcomes are possible.
   b. Construct a tree diagram to list all possible outcomes.
3. Jamie has decided to buy either a Mega or a Better Byte desktop computer. She also wants to purchase either Big Word, Word World, or Great Word word-processing software and either Big Number or Number World spreadsheet software.
   a. Use the Fundamental Principle of Counting to determine how many different packages of a computer and software Jamie has to choose from.
   b. Construct a tree diagram to list all possible packages of a computer and software.
4. Sammy's Sandwich Shop offers a soup, sandwich, and beverage combination at a special price. There are three sandwiches (turkey, tuna, and tofu), two soups (minestrone and split pea), and three beverages (coffee, milk, and mineral water) to choose from.
   a. Use the Fundamental Principle of Counting to determine how many different meal combinations are possible.
   b. Construct a tree diagram to list all possible soup, sandwich, and beverage combinations.
5. If you buy three pairs of jeans, four sweaters, and two pairs of boots, how many new outfits (consisting of a new pair of jeans, a new sweater, and a new pair of boots) will you have?
6. A certain model of automobile is available in six exterior colors, three interior colors, and three interior styles. In addition, the transmission can be either manual or automatic, and the engine can have either four or six cylinders. How many different versions of the automobile can be ordered?
7. To fulfill certain requirements for a degree, a student must take one course each from the following groups: health, civics, critical thinking, and elective. If there are four health, three civics, six critical thinking, and ten elective courses, how many different options for fulfilling the requirements does a student have?
8. To fulfill a requirement for a literature class, a student must read one short story by each of the following authors: Stephen King, Clive Barker, Edgar Allan Poe, and H. P. Lovecraft. If there are twelve King, six Barker, eight Poe, and eight Lovecraft stories to choose from, how many different combinations of reading assignments can a student choose from to fulfill the reading requirement?
9. A sporting goods store has fourteen lines of snow skis, seven types of bindings, nine types of boots, and three types of poles. Assuming that all items are compatible with each other, how many different complete ski equipment packages are available?
10. An audio equipment store has ten different amplifiers, four tuners, six turntables, eight tape decks, six compact disc players, and thirteen speakers. Assuming that all components are compatible with each other, how many different complete stereo systems are available?
11. A cafeteria offers a complete dinner that includes one serving each of appetizer, soup, entrée, and dessert for $6.99. If the menu has three appetizers, four soups, six entrées, and three desserts, how many different meals are possible?
12. A sandwich shop offers a "U-Chooz" special consisting of your choice of bread, meat, cheese, and special sauce (one each). If there are six different breads, eight meats, five cheeses, and four special sauces, how many different sandwiches are possible?
13. How many different Social Security numbers are possible? (A Social Security number consists of nine digits that can be repeated.)
14. To use an automated teller machine (ATM), a customer must enter his or her four-digit Personal Identification Number (PIN). How many different PINs are possible?
15. Every book published has an International Standard Book Number (ISBN). The number is a code used to identify the specific book and is of the form X-XXX-XXXXX-X, where X is one of digits 0, 1, 2, . . . , 9. How many different ISBNs are possible?
16. How many different Zip Codes are possible using (a) the old style (five digits) and (b) the new style (nine digits)? Why do you think the U.S. Postal Service introduced the new system?
17. Telephone area codes are three-digit numbers of the form XXX.
    a. Originally, the first and third digits were neither 0 nor 1 and the second digit was always a 0 or a 1. How many three-digit numbers of this type are possible?
    b. Over time, the restrictions listed in part (a) have been altered; currently, the only requirement is that the first digit is neither 0 nor 1. How many three-digit numbers of this type are possible?

c. Why were the original restrictions listed in part (a) altered?

18. Major credit cards such as VISA and MasterCard have a sixteen-digit account number of the form XXXX-XXXX-XXXX-XXXX. How many different numbers of this type are possible?

19. The serial number on a dollar bill consists of a letter followed by eight digits and then a letter. How many different serial numbers are possible, given the following conditions?

a. Letters and digits cannot be repeated.

b. Letters and digits can be repeated.

c. The letters are nonrepeated consonants and the digits can be repeated.

20. The serial number on a new twenty-dollar bill consists of two letters followed by eight digits and then a letter. How many different serial numbers are possible, given the following conditions?

a. Letters and digits cannot be repeated.

b. Letters and digits can be repeated.

c. The first and last letters are repeatable vowels, the second letter is a consonant, and the digits can be repeated.

21. Each student at State University has a student I.D. number consisting of four digits (the first digit is nonzero, and digits may be repeated) followed by three of the letters A, B, C, D, and E (letters may not be repeated). How many different student numbers are possible?

22. Each student at State College has a student I.D. number consisting of five digits (the first digit is nonzero, and digits may be repeated) followed by two of the letters A, B, C, D, and E (letters may not be repeated). How many different student numbers are possible?

In Exercises 23–38, find the indicated value.

23. 4! **24.** 5!

25. 10! **26.** 8!

27. 20! **28.** 25!

29. $6! \cdot 4!$ **30.** $8! \cdot 6!$

31. **a.** $\frac{6!}{4!}$ **b.** $\frac{6!}{2!}$ **32.** **a.** $\frac{8!}{6!}$ **b.** $\frac{8!}{2!}$

33. $\frac{8!}{5! \cdot 3!}$ **34.** $\frac{9!}{5! \cdot 4!}$

35. $\frac{8!}{4! \cdot 4!}$ **36.** $\frac{6!}{3! \cdot 3!}$

37. $\frac{82!}{80! \cdot 2!}$ **38.** $\frac{77!}{74! \cdot 3!}$

39. Find the value of $\frac{n!}{(n-r)!}$ when $n = 16$ and $r = 14$.

40. Find the value of $\frac{n!}{(n-r)!}$ when $n = 19$ and $r = 16$.

41. Find the value of $\frac{n!}{(n-r)!}$ when $n = 5$ and $r = 5$.

42. Find the value of $\frac{n!}{(n-r)!}$ when $n = r$.

43. Find the value of $\frac{n!}{(n-r)!r!}$ when $n = 7$ and $r = 3$.

44. Find the value of $\frac{n!}{(n-r)!r!}$ when $n = 7$ and $r = 4$.

45. Find the value of $\frac{n!}{(n-r)!r!}$ when $n = 5$ and $r = 5$.

46. Find the value of $\frac{n!}{(n-r)!r!}$ when $n = r$.

Answer the following questions using complete sentences and your own words.

• CONCEPT QUESTIONS

47. What is the Fundamental Principle of Counting? When is it used?

48. What is a factorial?

• HISTORY QUESTIONS

49. Who invented the modern symbol denoting a factorial? What symbol did it replace? Why?

THE NEXT LEVEL

If a person wants to pursue an advanced degree (something beyond a bachelor's or four-year degree), chances are the person must take a standardized exam to gain admission to a school or to be admitted into a specific program. These exams are intended to measure verbal, quantitative, and analytical skills that have developed throughout a person's life. Many classes and study guides are available to help people prepare for the exams. The following questions are typical of those found in the study guides.

Exercises 50–54 refer to the following: In an executive parking lot, there are six parking spaces in a row, labeled 1 through 6. Exactly five cars of five different colors—black, gray, pink, white, and yellow—are to be parked in the spaces. The cars can park in any of the spaces as long as the following conditions are met:

- The pink car must be parked in space 3.
- The black car must be parked in a space next to the space in which the yellow car is parked.
- The gray car cannot be parked in a space next to the space in which the white car is parked.

50. If the yellow car is parked in space 1, how many acceptable parking arrangements are there for the five cars?

a. 1 **b.** 2 **c.** 3 **d.** 4 **e.** 5

51. Which of the following must be true of any acceptable parking arrangement?

a. One of the cars is parked in space 2.

b. One of the cars is parked in space 6.

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c. There is an empty space next to the space in which the gray car is parked.
d. There is an empty space next to the space in which the yellow car is parked.
e. Either the black car or the yellow car is parked in a space next to space 3.

52. If the gray car is parked in space 2, none of the cars can be parked in which space?
a. 1 **b.** 3 **c.** 4 **d.** 5 **e.** 6

53. The white car could be parked in any of the spaces except which of the following?
a. 1 **b.** 2 **c.** 4 **d.** 5 **e.** 6

54. If the yellow car is parked in space 2, which of the following must be true?
a. None of the cars is parked in space 5.
b. The gray car is parked in space 6.
c. The black car is parked in a space next to the space in which the white car is parked.
d. The white car is parked in a space next to the space in which the pink car is parked.
e. The gray car is parked in a space next to the space in which the black car is parked.

2.4 Permutations and Combinations

Objectives

- Develop and apply the Permutation Formula
- Develop and apply the Combination Formula
- Determine the number of distinguishable permutations

The Fundamental Principle of Counting allows us to determine the total number of possible outcomes when a series of decisions (making selections from various categories) must be made. In Section 2.3, the examples and exercises involved selecting *one item each* from various categories; if you buy two pairs of jeans, three shirts, and two pairs of shoes, you will have twelve ($2 \cdot 3 \cdot 2 = 12$) new outfits (consisting of a new pair of jeans, a new shirt, and a new pair of shoes). In this section, we examine the situation when *more than one* item is selected from a category. If more than one item is selected, the selections can be made either *with* or *without* replacement.

With versus Without Replacement

Selecting items *with replacement* means that the same item *can* be selected more than once; after a specific item has been chosen, it is put back into the pool of future choices. Selecting items *without replacement* means that the same item *cannot* be selected more than once; after a specific item has been chosen, it is not replaced.

Suppose you must select a four-digit Personal Identification Number (PIN) for a bank account. In this case, the digits are selected with replacement; each time a specific digit is selected, the digit is put back into the pool of choices for the next selection. (Your PIN can be 3666; the same digit can be selected more than once.) When items are selected with replacement, we use the Fundamental Principle of Counting to determine the total number of possible outcomes; there are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ possible four-digit PINs.

In many situations, items cannot be selected more than once. For instance, when selecting a committee of three people from a group of twenty, you cannot select the same person more than once. Once you have selected a specific person (say, Lauren), you do not put her back into the pool of choices. When selecting items without replacement, depending on whether the order of selection is important, *permutations* or *combinations* are used to determine the total number of possible outcomes.

Permutations

When more than one item is selected (without replacement) from a single category, and the order of selection *is* important, the various possible outcomes are called **permutations.** For example, when the rankings (first, second, and third place) in a talent contest are announced, the order of selection is important; Monte in first, Lynn in second, and Ginny in third place is different from Ginny in first, Monte in second, and Lynn in third. "Monte, Lynn, Ginny" and "Ginny, Monte, Lynn" are different permutations of the contestants. Naturally, these selections are made without replacement; we cannot select Monte for first place and reselect him for second place.

EXAMPLE 1

FINDING THE NUMBER OF PERMUTATIONS Six local bands have volunteered to perform at a benefit concert, but there is enough time for only four bands to play. There is also some concern over the order in which the chosen bands will perform. How many different lineups are possible?

SOLUTION

We must select four of the six bands and put them in a specific order. The bands are selected without replacement; a band cannot be selected to play and then be reselected to play again. Because we must make four decisions, we draw four boxes and put the number of choices for each decision in each appropriate box. There are six choices for the opening band. Naturally, the opening band could not be the follow-up act, so there are only five choices for the next group. Similarly, there are four candidates for the third group and three choices for the closing band. The total number of different lineups possible is found by multiplying the number of choices for each decision:

$$\underset{\textbf{opening band}}{\boxed{6}} \times \boxed{5} \times \boxed{4} \times \underset{\textbf{closing band}}{\boxed{3}} = 360$$

With four out of six bands playing in the performance, 360 lineups are possible. *Because the order of selecting the bands is important, the various possible outcomes, or lineups, are called permutations;* there are 360 permutations of six items when the items are selected four at a time.

The computation in Example 1 is similar to a factorial, but the factors do not go all the way down to 1; the product $6 \cdot 5 \cdot 4 \cdot 3$ is a "truncated" (cut-off) factorial. We can change this truncated factorial into a complete factorial in the following manner:

$$\begin{aligned} 6 \cdot 5 \cdot 4 \cdot 3 &= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot (2 \cdot 1)}{(2 \cdot 1)} \quad \textbf{multiplying by } \frac{2}{2} \textbf{ and } \frac{1}{1} \\ &= \frac{6!}{2!} \end{aligned}$$

Notice that this last expression can be written as $\frac{6!}{2!} = \frac{6!}{(6-4)!}$. (Recall that we were selecting four out of six bands.) This result is generalized as follows.

PERMUTATION FORMULA

The number of **permutations,** or arrangements, of r items selected without replacement from a pool of n items ($r \leq n$), denoted by $_nP_r$, is

$$_nP_r = \frac{n!}{(n-r)!}$$

Permutations are used whenever more than one item is selected (without replacement) from a category and the order of selection is important.

Using the notation above and referring to Example 1, we note that 360 possible lineups of four bands selected from a pool of six can be denoted by $_6P_4 = \frac{6!}{(6-4)!} = 360$. Other notations can be used to represent the number of permutations of a group of items. In particular, the notations $_nP_r$, $P(n, r)$, P^n_r, and $P_{n,r}$ all represent the number of possible permutations (or arrangements) of r items selected (without replacement) from a pool of n items.

EXAMPLE 2

FINDING THE NUMBER OF PERMUTATIONS Three door prizes (first, second, and third) are to be awarded at a ten-year high school reunion. Each of the 112 attendees puts his or her name in a hat. The first name drawn wins a two-night stay at the Chat 'n' Rest Motel, the second name wins dinner for two at Juju's Kitsch-Inn, and the third wins a pair of engraved mugs. In how many different ways can the prizes be awarded?

SOLUTION

We must select 3 out of 112 people (without replacement), and the order in which they are selected *is* important. (Winning dinner is different from winning the mugs.) Hence, we must find the number of permutations of 3 items selected from a pool of 112:

$$\begin{aligned} _{112}P_3 &= \frac{112!}{(112-3)!} \\ &= \frac{112!}{109!} \\ &= \frac{112 \cdot 111 \cdot 110 \cdot 109 \cdot 108 \cdot \cdots \cdot 2 \cdot 1}{109 \cdot 108 \cdot \cdots \cdot 2 \cdot 1} \\ &= 112 \cdot 111 \cdot 110 \\ &= 1{,}367{,}520 \end{aligned}$$

There are 1,367,520 different ways in which the three prizes can be awarded to the 112 people.

In Example 2, if you try to use a calculator to find $\frac{112!}{109!}$ directly, you will not obtain an answer. Entering 112 and pressing $\boxed{x!}$ results in a calculator error. (Try it.) Because factorials get very large very quickly, most calculators are not able to find any factorial over 69!. (69! = $1.711224524 \times 10^{98}$.)

EXAMPLE 3

FINDING THE NUMBER OF PERMUTATIONS A bowling league has ten teams. In how many different ways can the teams be ranked in the standings at the end of a tournament? (Ties are not allowed.)

SOLUTION

Because order is important, we find the number of permutations of ten items selected from a pool of ten items:

$$
\begin{aligned}
{}_{10}P_{10} &= \frac{10!}{(10-10)!} \\
&= \frac{10!}{0!} \qquad \textbf{Recall that } 0! = 1. \\
&= \frac{10!}{1} \\
&= 3{,}628{,}800
\end{aligned}
$$

In a league containing ten teams, 3,628,800 different standings are possible at the end of a tournament.

Combinations

When items are selected from a group, the order of selection may or may not be important. If the order is important (as in Examples 1, 2, and 3), permutations are used to determine the total number of selections possible. What if the order of selection is *not* important? When more than one item is selected (without replacement) from a single category and the order of selection is not important, the various possible outcomes are called **combinations.**

EXAMPLE 4

LISTING ALL POSSIBLE COMBINATIONS Two adults are needed to chaperone a daycare center's field trip. Marcus, Vivian, Frank, and Keiko are the four managers of the center. How many different groups of chaperones are possible?

SOLUTION

In selecting the chaperones, the order of selection is *not* important; "Marcus and Vivian" is the same as "Vivian and Marcus." Hence, the permutation formula cannot be used. Because we do not yet have a shortcut for finding the total number of possibilities when the order of selection is not important, we must list all the possibilities:

Marcus and Vivian	Marcus and Frank	Marcus and Keiko
Vivian and Frank	Vivian and Keiko	Frank and Keiko

Therefore, six different groups of two chaperones are possible from the group of four managers. Because the order in which the people are selected is not important, the various possible outcomes, or groups of chaperones, are called *combinations;* there are six combinations when two items are selected from a pool of four.

Just as ${}_nP_r$ denotes the number of *permutations* of r elements selected from a pool of n elements, ${}_nC_r$ denotes the number of *combinations* of r elements selected from a pool of n elements. In Example 4, we found that there are six combinations of two people selected from a pool of four by listing all six of the combinations; that is, ${}_4C_2 = 6$. If we had a larger pool, listing each combination to find out how many there are would be extremely time consuming and tedious! Instead of listing, we take a different approach. We first find the number of permutations (with the permutation formula) and then alter that number to account for the distinction between permutations and combinations.

To find the number of combinations of two people selected from a pool of four, we first find the number of permutations:

$$ {}_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12 $$

This figure of 12 must be altered to account for the distinction between permutations and combinations.

In Example 4, we listed combinations; one such combination was "Marcus and Vivian." If we had listed permutations, we would have had to list both "Marcus and Vivian" and "Vivian and Marcus," because the *order* of selection matters with permutations. In fact, each combination of two chaperones listed in Example 4 generates two permutations; each pair of chaperones can be given in two different orders. Thus, there are twice as many permutations of two people selected from a pool of four as there are combinations. Alternatively, there are half as many combinations of two people selected from a pool of four as there are permutations. We used the permutation formula to find that ${}_4P_2 = 12$; thus,

$$ {}_4C_2 = \frac{1}{2} \cdot {}_4P_2 = \frac{1}{2}(12) = 6 $$

This answer certainly fits with Example 4; we listed exactly six combinations.

What if three of the four managers were needed to chaperone the daycare center's field trip? Rather than finding the number of combinations by listing each possibility, we first find the number of permutations and then alter that number to account for the distinction between permutations and combinations.

The number of permutations of three people selected from a pool of four is

$$ {}_4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24 $$

We know that some of these permutations represent the same combination. For example, the combination "Marcus and Vivian and Keiko" generates $3! = 6$ different permutations (using initials, they are: MVK, MKV, KMV, KVM, VMK, VKM). Because each combination of three people generates six different permutations, there are one-sixth as many combinations as permutations. Thus,

$$ {}_4C_3 = \frac{1}{6} \cdot {}_4P_3 = \frac{1}{6}(24) = 4 $$

This means that if three of the four managers were needed to chaperone the daycare center's field trip, there would be ${}_4C_3 = 4$ possible combinations.

We just saw that when two items are selected from a pool of n items, each combination of two generates $2! = 2$ permutations, so

$$ {}_nC_2 = \frac{1}{2!} \cdot {}_nP_2 $$

We also saw that when three items are selected from a pool of n items, each combination of three generates $3! = 6$ permutations, so

$$ {}_nC_3 = \frac{1}{3!} \cdot {}_nP_3 $$

More generally, when r items are selected from a pool of n items, each combination of r items generates $r!$ permutations, so

$$
\begin{aligned}
{}_nC_r &= \frac{1}{r!} \cdot {}_nP_r \\
&= \frac{1}{r!} \cdot \frac{n!}{(n-r)!} && \text{using the Permutation Formula} \\
&= \frac{n!}{r! \cdot (n-r)!} && \text{multiplying the fractions together}
\end{aligned}
$$

COMBINATION FORMULA

The number of distinct **combinations** of r items selected without replacement from a pool of n items ($r \le n$), denoted by ${}_nC_r$, is

$$
{}_nC_r = \frac{n!}{(n-r)!\, r!}
$$

Combinations are used whenever one or more items are selected (without replacement) from a category and the order of selection is not important.

EXAMPLE 5

FINDING THE NUMBER OF COMBINATIONS A DVD club sends you a brochure that offers any five DVDs from a group of fifty of today's hottest releases. How many different selections can you make?

SOLUTION

Because the order of selection is *not* important, we find the number of combinations when five items are selected from a pool of fifty:

$$
\begin{aligned}
{}_{50}C_5 &= \frac{50!}{(50-5)!\, 5!} \\
&= \frac{50!}{45!\, 5!} \\
&= \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= 2{,}118{,}760
\end{aligned}
$$

50 $\boxed{x!}$ $\boxed{\div}$ $\boxed{(}$ 45 $\boxed{x!}$ $\boxed{\times}$ 5 $\boxed{x!}$ $\boxed{)}$ $\boxed{=}$

Graphing calculators have buttons that will calculate ${}_nP_r$ and ${}_nC_r$. To use them, do the following:

- Type the value of n. (For example, type the number 50.)
- Press the $\boxed{\text{MATH}}$ button.
- Press the right arrow button $\boxed{\rightarrow}$ as many times as necessary to highlight $\boxed{\text{PRB}}$.
- Press the down arrow button $\boxed{\downarrow}$ as many times as necessary to highlight the appropriate symbol—$\boxed{{}_nP_r}$ for permutations, $\boxed{{}_nC_r}$ for combinations—and press $\boxed{\text{ENTER}}$.
- Type the value of r. (For example, type the number 5.)
- Press $\boxed{\text{ENTER}}$ to execute the calculation.

On a Casio graphing calculator, do the following:

- Press the $\boxed{\text{MENU}}$ button; this gives you access to the main menu.
- Press 1 to select the RUN mode; this mode is used to perform arithmetic operations.
- Type the value of *n*. (For example, type the number 50.)
- Press the $\boxed{\text{OPTN}}$ button; this gives you access to various options displayed at the bottom of the screen.
- Press the $\boxed{\text{F6}}$ button to see more options (i.e., $\boxed{\rightarrow}$).
- Press the $\boxed{\text{F3}}$ button to select probability options (i.e., $\boxed{\text{PROB}}$).
- Press the $\boxed{\text{F3}}$ button to select combinations (i.e., $\boxed{{}_nC_r}$) or the $\boxed{\text{F2}}$ button to select permutations (i.e., $\boxed{{}_nP_r}$).
- Type the value of *r*. (For example, type the number 5.)
- Press the $\boxed{\text{EXE}}$ button to execute the calculation.

50 $\boxed{\text{MATH}}$ $\boxed{\text{PRB}}$ $\boxed{{}_nC_r}$ 5 $\boxed{\text{ENTER}}$

Casio 50 $\boxed{\text{OPTN}}$ $\boxed{\rightarrow}$ (i.e., $\boxed{\text{F6}}$) $\boxed{\text{PROB}}$ (i.e., $\boxed{\text{F3}}$) $\boxed{\text{nCr}}$ (i.e., $\boxed{\text{F3}}$) 5 $\boxed{\text{EXE}}$

In choosing five out of fifty DVDs, 2,118,760 combinations are possible.

EXAMPLE 6

FINDING THE NUMBER OF COMBINATIONS A group consisting of twelve women and nine men must select a five-person committee. How many different committees are possible if it must consist of the following?

a. three women and two men **b.** any mixture of men and women

SOLUTION

a. Our problem involves two categories: women and men. The Fundamental Principle of Counting tells us to draw two boxes (one for each category), enter the number of choices for each, and multiply:

$$\boxed{\begin{array}{c}\text{the number of ways}\\ \text{in which we can}\\ \text{select three out}\\ \text{of twelve}\\ \text{women}\end{array}} \times \boxed{\begin{array}{c}\text{the number of ways}\\ \text{in which we can}\\ \text{select two out}\\ \text{of nine}\\ \text{men}\end{array}} = ?$$

Because the order of selecting the members of a committee is not important, we will use combinations:

$$\begin{aligned}({}_{12}C_3)\cdot({}_9C_2) &= \frac{12!}{(12-3)!\cdot 3!}\cdot\frac{9!}{(9-2)!\cdot 2!}\\ &= \frac{12!}{9!\cdot 3!}\cdot\frac{9!}{7!\cdot 2!}\\ &= \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}\cdot\frac{9\cdot 8}{2\cdot 1}\\ &= 220\cdot 36\\ &= 7{,}920\end{aligned}$$

(12 x! ÷ (9 x! × 3 x!)) × (9 x! ÷
(7 x! × 2 x!)) =

12 MATH PRB nCr 3 × 9 MATH PRB nCr 2 ENTER

There are 7,920 different committees consisting of three women and two men.

b. Because the gender of the committee members doesn't matter, our problem involves only one category: people. We must choose five out of the twenty-one people, and the order of selection is not important:

$$\begin{aligned} {}_{21}C_5 &= \frac{21!}{(21-5)! \cdot 5!} \\ &= \frac{21!}{16! \cdot 5!} \\ &= \frac{21 \cdot 20 \cdot 19 \cdot 18 \cdot 17}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 20{,}349 \end{aligned}$$

21 x! ÷ (16 x! × 5 x!) =

21 MATH PRB nCr 5 ENTER

There are 20,349 different committees consisting of five people.

EXAMPLE 7

EVALUATING THE COMBINATION FORMULA Find the value of ${}_5C_r$ for the following values of r:

a. $r = 0$ **b.** $r = 1$ **c.** $r = 2$ **d.** $r = 3$ **e.** $r = 4$ **f.** $r = 5$

SOLUTION

a. $${}_5C_0 = \frac{5!}{(5-0)! \cdot 0!} = \frac{5!}{5! \cdot 0!} = 1$$

b. $${}_5C_1 = \frac{5!}{(5-1)! \cdot 1!} = \frac{5!}{4! \cdot 1!} = 5$$

c. $${}_5C_2 = \frac{5!}{(5-2)! \cdot 2!} = \frac{5!}{3! \cdot 2!} = 10$$

d. $${}_5C_3 = \frac{5!}{(5-3)! \cdot 3!} = \frac{5!}{2! \cdot 3!} = 10$$

e. $${}_5C_4 = \frac{5!}{(5-4)! \cdot 4!} = \frac{5!}{1! \cdot 4!} = 5$$

f. $${}_5C_5 = \frac{5!}{(5-5)! \cdot 5!} = \frac{5!}{0! \cdot 5!} = 1$$

The combinations generated in Example 7 exhibit a curious pattern. Notice that the values of ${}_5C_r$ are symmetric: ${}_5C_0 = {}_5C_5$, ${}_5C_1 = {}_5C_4$, and ${}_5C_2 = {}_5C_3$. Now examine the diagram in Figure 2.41. Each number in this "triangle" of numbers is the sum of two numbers in the row immediately above it. For example, $2 = 1 + 1$ and $10 = 4 + 6$, as shown by the inserted arrows. It is no coincidence that the values of ${}_5C_r$ found in Example 7 also appear as a row of numbers in this "magic" triangle. In fact, the sixth row contains all the values of ${}_5C_r$ for $r = 0, 1, 2, 3, 4,$ and 5. In general, the $(n + 1)^{\text{th}}$ row of the triangle contains all the values of ${}_nC_r$ for $r = 0, 1, 2, \ldots, n$; alternatively, the n^{th} row of the triangle contains all the values of ${}_{n-1}C_r$ for $r = 0, 1, 2, \ldots, n-1$. For example, the values of ${}_9C_r$, for $r = 0, 1, 2, \ldots, 9$, are in the tenth row, and vice versa, the entries in the tenth row are the values of ${}_9C_r$, for $r = 0, 1, 2, \ldots, 9$.

1st row						1					
2nd row					1		1				
3rd row				1		2		1			
4th row			1		3		3		1		
5th row		1		4		6		4		1	
6th row	1		5		10		10		5		1

and so on

FIGURE 2.41 Pascal's triangle.

Historically, this triangular pattern of numbers is referred to as *Pascal's Triangle*, in honor of the French mathematician, scientist, and philosopher Blaise Pascal (1623–1662). Pascal is a cofounder of probability theory (see the Historical Note in Section 3.1). Although the triangle has Pascal's name attached to it, this "magic" arrangement of numbers was known to other cultures hundreds of years before Pascal's time.

The most important part of any problem involving combinatorics is deciding which counting technique (or techniques) to use. The following list of general steps and the flowchart in Figure 2.42 can help you to decide which method or methods to use in a specific problem.

WHICH COUNTING TECHNIQUE?

1. What is being selected?
2. If the selected items can be repeated, use the **Fundamental Principle of Counting** and multiply the number of choices for each category.
3. If there is only one category, use:
   combinations if the order of selection does not matter—that is, r items can be selected from a pool of n items in ${}_nC_r = \frac{n!}{(n-r)!\cdot r!}$ ways.
   permutations if the order of selection does matter—that is, r items can be selected from a pool of n items in ${}_nP_r = \frac{n!}{(n-r)!}$ ways.
4. If there is more than one category, use the **Fundamental Principle of Counting** with one box per category.
   a. If you are selecting one item per category, the number in the box for that category is the number of choices for that category.
   b. If you are selecting more than one item per category, the number in the box for that category is found by using step 3.

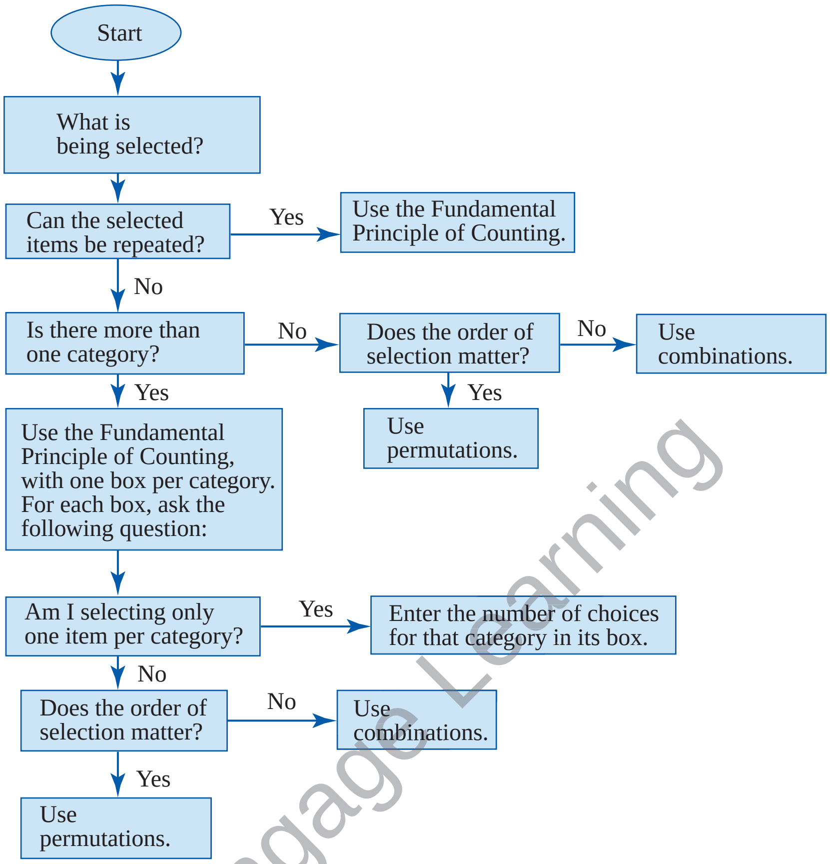


FIGURE 2.42 Which counting technique?

EXAMPLE 8

USING THE "WHICH COUNTING TECHNIQUE?" FLOWCHART A standard deck of playing cards contains fifty-two cards.

a. How many different five-card hands containing four kings are possible?
b. How many different five-card hands containing four queens are possible?
c. How many different five-card hands containing four kings or four queens are possible?
d. How many different five-card hands containing four of a kind are possible?

SOLUTION

a. We use the flowchart in Figure 2.42 and answer the following questions.

Q. What is being selected?
A. Playing cards.

Q. Can the selected items be repeated?
A. No.

Q. Is there more than one category?
A. Yes: Because we must have five cards, we need four kings and one non-king. Therefore, we need two boxes:

$$\boxed{\text{kings}} \times \boxed{\text{non-kings}}$$

Q. Am I selecting only one item per category?
A. *Kings:* no. Does the order of selection matter? No: Use combinations. Because there are $n = 4$ kings in the deck and we want to select $r = 4$, we must compute ${}_4C_4$. *Non-kings:* yes. Enter the number of choices for that category: There are 48 non-kings.

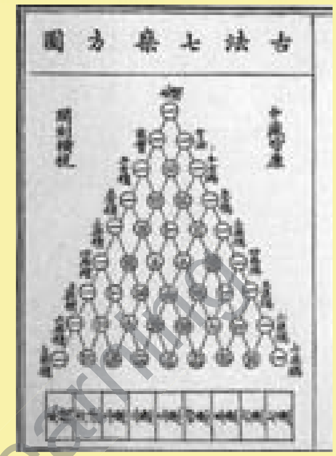
HISTORICAL NOTE

CHU SHIH-CHIEH, CIRCA 1280–1303

Chu Shih-chieh was the last and most acclaimed mathematician of the Sung Dynasty in China. Little is known of his personal life; the actual dates of his birth and death are unknown. His work appears to have flourished during the close of the thirteenth century. It is believed that Chu Shih-chieh spent many years as a wandering scholar, earning a living by teaching mathematics to those who wanted to learn.

Two of Chu Shih-chieh's works have survived the centuries. The first, *Suan-hsüeh ch'i-meng (Introduction to Mathematical Studies)*, was written in 1299 and contains elementary mathematics. This work was very influential in Japan and Korea, although it was lost in China until the nineteenth century. Written in 1303, Chu's second work *Ssu-yüan yü-chien (Precious Mirror of the Four Elements)* contains more advanced mathematics. The topics of *Precious Mirror* include the solving of simultaneous equations and the solving of equations up to the fourteenth degree.

Of the many diagrams in *Precious Mirror,* one has special interest: the arithmetic triangle. Chu Shih-chieh's triangle contains the first eight rows of what is known in the West as Pascal's Triangle. However, Chu does not claim credit for the triangle; he refers to it as "a diagram of the *old* method for finding eighth and lower powers." "Pascal's" Triangle was known to the Chinese well over 300 years before Pascal was born!



The "Pascal" Triangle as depicted in 1303 at the front of Chu Shih-chieh's Ssu-yüan yü-chien. *It is entitled "The Old Method Chart of the Seven Multiplying Squares" and tabulates the binomial coefficients up to the eighth power.*

$$\boxed{\text{kings}} \times \boxed{\text{non-kings}} = \boxed{{}_4C_4} \times \boxed{48}$$
$$= \frac{4!}{(4-4)! \cdot 4!} \cdot 48$$
$$= \frac{4!}{0! \cdot 4!} \cdot 48$$
$$= 1 \cdot 48$$
$$= 48$$

There are forty-eight different five-card hands containing four kings.

b. Using the same method as in part (a), we would find that there are forty-eight different five-card hands containing four queens; the number of five-card hands containing four queens is the same as the number of five-card hands containing four kings.

c. To find the number of five-card hands containing four kings or four queens, we define the following sets:

$$A = \{\text{five-card hands} \mid \text{the hand contains four kings}\}$$
$$B = \{\text{five-card hands} \mid \text{the hand contains four queens}\}$$

Consequently,

$$A \cup B = \{\text{five-card hands} \mid \text{the hand contains four kings } \textit{or} \text{ four queens}\}$$
$$A \cap B = \{\text{five-card hands} \mid \text{the hand contains four kings } \textit{and} \text{ four queens}\}$$

(Recall that the union symbol, $\cup$, may be interpreted as the word "or," while the intersection symbol, $\cap$, may be interpreted as the word "and." See Figure 2.13 for a comparison of set theory and logic.)

Because there are no five-card hands that contain four kings *and* four queens, we note that $n(A \cap B) = 0$.

Using the Union/Intersection Formula for the Union of Sets, we obtain

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 48 + 48 - 0 \\ &= 96 \end{aligned}$$

There are ninety-six different five-card hands containing four kings or four queens.

d. *Four of a kind* means four cards of the same "denomination," that is, four 2's, or four 3's, or four 4's, or . . . , or four kings, or four aces. Now, regardless of the denomination of the card, there are forty-eight different five-card hands that contain four of any specific denomination; there are forty-eight different five-card hands that contain four 2's, there are forty-eight different five-card hands that contain four 3's, there are forty-eight different five-card hands that contain four 4's, and so on. As is shown in part (c), the word "or" implies that we *add* cardinal numbers. Consequently,

$$\begin{aligned} n(\text{four of a kind}) &= n(\text{four 2's or four 3's or . . . or four kings or four aces}) \\ &= n(\text{four 2's}) + n(\text{four 3's}) + \cdots + n(\text{four kings}) \\ &\quad + n(\text{four aces}) \\ &= 48 + 48 + \cdots + 48 + 48 \quad \text{(thirteen times)} \\ &= 13 \times 48 \\ &= 624 \end{aligned}$$

There are 624 different five-card hands containing four of a kind.

As is shown in Example 8, there are 624 possible five-card hands that contain four of a kind. When you are dealt five cards, what is the likelihood (or probability) that you will receive one of these hands? This question, and its answer, will be explored in Section 3.4, "Combinatorics and Probability."

Permutations of Identical Items

In how many different ways can the three letters in the word "SAW" be arranged? As we know, arrangements are referred to as *permutations,* so we can apply the Permutation Formula, ${}_nP_r = \frac{n!}{(n-r)!}$.

Therefore,

$${}_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3 \cdot 2 \cdot 1}{1} = 6$$

The six permutations of the letters in SAW are

SAW SWA AWS ASW WAS WSA

In general, if we have three *different* items (the letters in SAW), we can arrange them in $3! = 6$ ways. However, this method applies only if the items are all different (distinct).

What happens if some of the items are the same (identical)? For example, in how many different ways can the three letters in the word "SEE" be arranged? Because two of the letters are identical (E), we cannot use the Permutation Formula directly; we take a slightly different approach. Temporarily, let us assume that the E's are written in different colored inks, say, red and blue. Therefore, SEE could be expressed as SEE. These three symbols could be arranged in $3! = 6$ ways as follows:

SEE SEE ESE ESE EES EES

If we now remove the color, the arrangements are

SEE SEE ESE ESE EES EES

Some of these arrangements are duplicates of others; as we can see, there are only three different or **distinguishable permutations,** namely, SEE, ESE, and EES. Notice that when $n = 3$ (the total number of letters in SEE) and $x = 2$ (the number of identical letters), we can divide $n!$ by $x!$ to obtain the number of distinguishable permutations; that is,

$$\frac{n!}{x!} = \frac{3!}{2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 3$$

This method is applicable because dividing by the factorial of the repeated letter eliminates the duplicate arrangements; the method may by generalized as follows.

DISTINGUISHABLE PERMUTATIONS OF IDENTICAL ITEMS

The number of **distinguishable permutations** (or arrangements) of n items in which x items are identical, y items are identical, z items are identical, and so on, is $\frac{n!}{x!y!z!\cdots}$. That is, to find the number of distinguishable permutations, divide the total factorial by the factorial of each repeated item.

EXAMPLE 9 FINDING THE NUMBER OF DISTINGUISHABLE PERMUTATIONS

Find the number of distinguishable permutations of the letters in the word "MISSISSIPPI."

SOLUTION

The word "MISSISSIPPI" has $n = 11$ letters; I is repeated $x = 4$ times, S is repeated $y = 4$ times, and P is repeated $z = 2$ times. Therefore, we divide the total factorial by the factorial of each repeated letter and obtain

$$\frac{n!}{x!y!z!} = \frac{11!}{4!4!2!} = 34{,}650$$

The letters in the word MISSISSIPPI can be arranged in 34,650 ways. (Note that if the 11 letters were all different, there would be $11! = 39{,}96{,}800$ permutations.)

2.4 EXERCISES

In Exercises 1–12, find the indicated value:

1. **a.** ${}_7P_3$ **b.** ${}_7C_3$
2. **a.** ${}_8P_4$ **b.** ${}_8C_4$
3. **a.** ${}_5P_5$ **b.** ${}_5C_5$
4. **a.** ${}_9P_0$ **b.** ${}_9C_0$
5. **a.** ${}_{14}P_1$ **b.** ${}_{14}C_1$
6. **a.** ${}_{13}C_3$ **b.** ${}_{13}C_{10}$
7. **a.** ${}_{100}P_3$ **b.** ${}_{100}C_3$
8. **a.** ${}_{80}P_4$ **b.** ${}_{80}C_4$
9. **a.** ${}_xP_{x-1}$ **b.** ${}_xC_{x-1}$
10. **a.** ${}_xP_1$ **b.** ${}_xC_1$
11. **a.** ${}_xP_2$ **b.** ${}_xC_2$
12. **a.** ${}_xP_{x-2}$ **b.** ${}_xC_{x-2}$
13. **a.** Find ${}_3P_2$.
    b. List all of the permutations of {a, b, c} when the elements are taken two at a time.
14. **a.** Find ${}_3C_2$.
    b. List all of the combinations of {a, b, c} when the elements are taken two at a time.
15. **a.** Find ${}_4C_2$.
    b. List all of the combinations of {a, b, c, d} when the elements are taken two at a time.
16. **a.** Find ${}_4P_2$.
    b. List all of the permutations of {a, b, c, d} when the elements are taken two at a time.
17. An art class consists of eleven students. All of them must present their portfolios and explain their work to the instructor and their classmates at the end of the semester.
    a. If their names are drawn from a hat to determine who goes first, second, and so on, how many presentation orders are possible?

   b. If their names are put in alphabetical order to determine who goes first, second, and so on, how many presentation orders are possible?

18. An English class consists of twenty-three students, and three are to be chosen to give speeches in a school competition. In how many different ways can the teacher choose the team, given the following conditions?

   a. The order of the speakers is important.

   b. The order of the speakers is not important.

19. In how many ways can the letters in the word "school" be arranged? (See the photograph below.)

20. A committee of four is to be selected from a group of sixteen people. How many different committees are possible, given the following conditions?

   a. There is no distinction between the responsibilities of the members.

   b. One person is the chair, and the rest are general members.

   c. One person is the chair, one person is the secretary, one person is responsible for refreshments, and one person cleans up after meetings.

21. A softball league has thirteen teams. If every team must play every other team once in the first round of league play, how many games must be scheduled?

22. In a group of eighteen people, each person shakes hands once with each other person in the group. How many handshakes will occur?

23. A softball league has thirteen teams. How many different end-of-the-season rankings of first, second, and third place are possible (disregarding ties)?

24. Three hundred people buy raffle tickets. Three winning tickets will be drawn at random.

   a. If first prize is \$100, second prize is \$50, and third prize is \$20, in how many different ways can the prizes be awarded?

   b. If each prize is \$50, in how many different ways can the prizes be awarded?

25. A group of nine women and six men must select a four-person committee. How many committees are possible if it must consist of the following?

   a. two women and two men

   b. any mixture of men and women

   c. a majority of women

26. A group of ten seniors, eight juniors, six sophomores, and five freshmen must select a committee of four. How many committees are possible if the committee must contain the following:

   a. one person from each class

   b. any mixture of the classes

   c. exactly two seniors

Exercises 27–32 refer to a deck of fifty-two playing cards (jokers not allowed). If you are unfamiliar with playing cards, see the end of Section 3.1 for a description of a standard deck.

27. How many five-card poker hands are possible?

28. **a.** How many five-card poker hands consisting of all hearts are possible?

   b. How many five-card poker hands consisting of all cards of the same suit are possible?

AP Photo/News & Record, Joseph Rodriguez

Exercise 19: Right letters, wrong order. SHCOOL is painted along the newly paved road leading to Southern Guilford High School on Drake Road Monday, August 9, 2010, in Greensboro, N.C.

29. **a.** How many five-card poker hands containing exactly three aces are possible?
b. How many five-card poker hands containing three of a kind are possible?

30. **a.** How many five-card poker hands consisting of three kings and two queens are possible?
b. How many five-card poker hands consisting of three of a kind and a pair (a *full house*) are possible?

31. How many five-card poker hands containing two pair are possible?

HINT: You must select two of the thirteen ranks, then select a pair of each, and then one of the remaining cards.

32. How many five-card poker hands containing exactly one pair are possible?

HINT: After selecting a pair, you must select three of the remaining twelve ranks and then select one card of each.

33. A 6/53 lottery requires choosing six of the numbers 1 through 53. How many different lottery tickets can you choose? (Order is not important, and the numbers do not repeat.)

34. A 7/39 lottery requires choosing seven of the numbers 1 through 39. How many different lottery tickets can you choose? (Order is not important, and the numbers do not repeat.)

35. A 5/36 lottery requires choosing five of the numbers 1 through 36. How many different lottery tickets can you choose? (Order is not important, and the numbers do not repeat.)

36. A 6/49 lottery requires choosing six of the numbers 1 through 49. How many different lottery tickets can you choose? (Order is not important, and the numbers do not repeat.)

37. Which lottery would be easier to win, a 6/53 or a 5/36? Why?

HINT: See Exercises 33 and 35.

38. Which lottery would be easier to win, a 6/49 or a 7/39? Why?

HINT: See Exercises 34 and 36.

39. **a.** Find the sum of the entries in the first row of Pascal's Triangle.
b. Find the sum of the entries in the second row of Pascal's Triangle.
c. Find the sum of the entries in the third row of Pascal's Triangle.
d. Find the sum of the entries in the fourth row of Pascal's Triangle.
e. Find the sum of the entries in the fifth row of Pascal's Triangle.
f. Is there a pattern to the answers to parts (a)–(e)? If so, describe the pattern you see.
g. Use the pattern described in part (f) to predict the sum of the entries in the sixth row of Pascal's triangle.
h. Find the sum of the entries in the sixth row of Pascal's Triangle. Was your prediction in part (g) correct?
i. Find the sum of the entries in the n^{th} row of Pascal's Triangle.

40. **a.** Add adjacent entries of the sixth row of Pascal's Triangle to obtain the seventh row.
b. Find ${}_6C_r$ for $r = 0, 1, 2, 3, 4, 5,$ and 6.
c. How are the answers to parts (a) and (b) related?

41. Use Pascal's Triangle to answer the following.
a. In which row would you find the value of ${}_4C_2$?
b. In which row would you find the value of ${}_nC_r$?
c. Is ${}_4C_2$ the second number in the fourth row?
d. Is ${}_4C_2$ the third number in the fifth row?
e. What is the location of ${}_nC_r$? Why?

42. Given the set $S = \{a, b, c, d\}$, answer the following.
a. How many one-element subsets does S have?
b. How many two-element subsets does S have?
c. How many three-element subsets does S have?
d. How many four-element subsets does S have?
e. How many zero-element subsets does S have?
f. How many subsets does S have?
g. If $n(S) = k$, how many subsets will S have?

In Exercises 43–50, find the number of permutations of the letters in each word.

43.	ALASKA	**44.**	ALABAMA
45.	ILLINOIS	**46.**	HAWAII
47.	INDIANA	**48.**	TENNESSEE
49.	TALLAHASSEE	**50.**	PHILADELPHIA

The words in each of Exercises 51–54 are homonyms *(words that are pronounced the same but have different meanings). Find the number of permutations of the letters in each word.*

51.	**a.** PIER	**b.** PEER
52.	**a.** HEAR	**b.** HERE
53.	**a.** STEAL	**b.** STEEL
54.	**a.** SHEAR	**b.** SHEER

Answer the following questions using complete sentences and your own words.

• CONCEPT QUESTIONS

55. Suppose you want to know how many ways r items can be selected from a group of n items. What determines whether you should calculate ${}_nP_r$ or ${}_nC_r$?

56. For any given values of n and r, which is larger, ${}_nP_r$ or ${}_nC_r$? Why?

THE NEXT LEVEL

If a person wants to pursue an advanced degree (something beyond a bachelor's or four-year degree), chances are the person must take a standardized exam to gain admission to a school or to be admitted into a specific program. These exams are intended to measure verbal, quantitative, and analytical skills that have developed throughout a person's life. Many classes and study guides are available to help people prepare for the exams. The following questions are typical of those found in the study guides.

Exercises 57–61 refer to the following: A baseball league has six teams: A, B, C, D, E, and F. All games are played at 7:30 P.M. on Fridays, and there are sufficient fields for each team to play a game every Friday night. Each team must play each other team exactly once, and the following conditions must be met:

- Team A plays team D first and team F second.
- Team B plays team E first and team C third.
- Team C plays team F first.

57. What is the total number of games that each team must play during the season?
a. 3 **b.** 4 **c.** 5 **d.** 6 **e.** 7

58. On the first Friday, which of the following pairs of teams play each other?
a. A and B; C and F; D and E
b. A and B; C and E; D and F
c. A and C; B and E; D and F
d. A and D; B and C; E and F
e. A and D; B and E; C and F

59. Which of the following teams must team B play second?
a. A **b.** C **c.** D **d.** E **e.** F

60. The last set of games could be between which teams?
a. A and B; C and F; D and E
b. A and C; B and F; D and E
c. A and D; B and C; E and F
d. A and E; B and C; D and F
e. A and F; B and E; C and D

61. If team D wins five games, which of the following must be true?
a. Team A loses five games.
b. Team A wins four games.
c. Team A wins its first game.
d. Team B wins five games.
e. Team B loses at least one game.

WEB PROJECT

62. Write a research paper on any historical topic referred to in this section or in a previous section. Following is a partial list of topics:
- John Venn
- Augustus De Morgan
- Chu Shih-chieh

Some useful links for this web project are listed on the text web site: **www.cengage.com/math/johnson**

2.5 Infinite Sets

OBJECTIVES

- Determine whether two sets are equivalent
- Establish a one-to-one correspondence between the elements of two sets
- Determine the cardinality of various infinite sets.

WARNING: Many leading nineteenth-century mathematicians and philosophers claim that the study of infinite sets may be dangerous to your mental health.

Consider the sets E and N, where $E = \{2, 4, 6, \ldots\}$ and $N = \{1, 2, 3, \ldots\}$. Both are examples of infinite sets (they "go on forever"). E is the set of all even counting numbers, and N is the set of all counting (or natural) numbers. Because every element of E is an element of N, E is a subset of N. In addition, N contains elements not in E; therefore, E is a *proper* subset of N. Which set is "bigger," E or N? Intuition might lead many people to think that N is twice as big as E because N contains all the even counting numbers *and* all the odd counting numbers. Not so! According to the work of Georg Cantor (considered by many to be the father of set theory), N and E have exactly the same number of elements! This seeming paradox, a *proper* subset that has the *same number* of elements as the set from which it came, caused a philosophic uproar in the late nineteenth century. (Hence, the warning at the beginning of this section.) To study Cantor's work (which is now accepted and considered a cornerstone in modern mathematics), we must first investigate the meaning of a one-to-one correspondence and equivalent sets.

One-to-One Correspondence

Is there any relationship between the sets $A = \{\text{one, two, three}\}$ and $B = \{\text{Pontiac, Chevrolet, Ford}\}$? Although the sets contain different types of things (numbers versus automobiles), each contains the same number of things; they are the same size. This relationship (being the same size) forms the basis of a one-to-one correspondence. A **one-to-one correspondence** between the sets A and B is a pairing up of the elements of A and B such that each element of A is paired up with exactly one element of B, and vice versa, with no element left out. For instance, the elements of A and B might be paired up as follows:

$$\begin{array}{ccc} \text{one} & \text{two} & \text{three} \\ \updownarrow & \updownarrow & \updownarrow \\ \text{Pontiac} & \text{Chevrolet} & \text{Ford} \end{array}$$

(Other correspondences, or matchups, are possible.) If two sets have the same cardinal number, their elements can be put into a one-to-one correspondence. Whenever a one-to-one correspondence exists between the elements of two sets A and B, the sets are **equivalent** (denoted by $A \sim B$). Hence, equivalent sets have the same number of elements.

If two sets have different cardinal numbers, it is not possible to construct a one-to-one correspondence between their elements. The sets $C = \{\text{one, two}\}$ and $B = \{\text{Pontiac, Chevrolet, Ford}\}$ do *not* have a one-to-one correspondence; no matter how their elements are paired up, one element of B will always be left over (B has more elements; it is "bigger"):

$$\begin{array}{ccc} \text{one} & \text{two} & \\ \updownarrow & \updownarrow & \\ \text{Pontiac} & \text{Chevrolet} & \text{Ford} \end{array}$$

The sets C and B are *not* equivalent.

Given two sets A and B, if any one of the following statements is true, then the other statements are also true:

1. There exists a one-to-one correspondence between the elements of A and B.
2. A and B are equivalent sets.
3. A and B have the same cardinal number; that is, $n(A) = n(B)$.

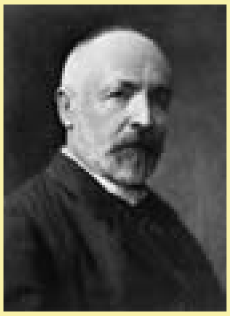
EXAMPLE 1

DETERMINING WHETHER TWO SETS ARE EQUIVALENT Determine whether the sets in each of the following pairs are equivalent. If they are equivalent, list a one-to-one correspondence between their elements.

a. $A = \{\text{John, Paul, George, Ringo}\}$;
$B = \{\text{Lennon, McCartney, Harrison, Starr}\}$
b. $C = \{\alpha, \beta, \chi, \delta\}$; $D = \{\text{I, O}, \Delta\}$
c. $A = \{1, 2, 3, \ldots, 48, 49, 50\}$; $B = \{1, 3, 5, \ldots, 95, 97, 99\}$

HISTORICAL NOTE

GEORG CANTOR, 1845–1918



Georg Ferdinand Ludwig Philip Cantor was born in St. Petersburg, Russia. His father was a stockbroker and wanted his son to become an engineer; his mother was an artist and musician. Several of Cantor's maternal relatives were accomplished musicians; in his later years, Cantor often wondered how his life would have turned out if he had become a violinist instead of pursuing a controversial career in mathematics.

Following his father's wishes, Cantor began his engineering studies at the University of Zurich in 1862. However, after one semester, he decided to study philosophy and pure mathematics. He transferred to the prestigious University of Berlin, studied under the famed mathematicians Karl Weierstrass, Ernst Kummer, and Leopold Kronecker, and received his doctorate in 1867. Two years later, Cantor accepted a teaching position at the University of Halle and remained there until he retired in 1913.

Cantor's treatises on set theory and the nature of infinite sets were first published in 1874 in *Crelle's Journal,* which was influential in mathematical circles. On their publication, Cantor's theories generated much controversy among mathematicians and philosophers. Paradoxes concerning the cardinal numbers of infinite sets, the nature of infinity, and Cantor's form of logic were unsettling to many, including Cantor's former teacher Leopold Kronecker. In fact, some felt that Cantor's work was not just revolutionary but actually dangerous. Kronecker led the attack on Cantor's theories. He was an editor of *Crelle's Journal* and held up the publication of one of Cantor's subsequent articles for so long that Cantor refused to publish ever again in the *Journal.* In addition, Kronecker blocked Cantor's efforts to obtain a teaching position at the University of Berlin. Even though Cantor was attacked by Kronecker and his followers, others respected him. Realizing the importance of communication among scholars, Cantor founded the Association of German Mathematicians in 1890 and served as its president for many years. In addition, Cantor was instrumental in organizing the first International Congress of Mathematicians, held in Zurich in 1897.

As a result of the repeated attacks on him and his work, Cantor suffered many nervous breakdowns, the first when he was thirty-nine. He died in a mental hospital in Halle at the age of seventy-three, never having received proper recognition for the true value of his discoveries. Modern mathematicians believe that Cantor's form of logic and his concepts of infinity revolutionized all of mathematics, and his work is now considered a cornerstone in its development.

SUR UNE PROPRIÉTÉ DU SYSTÈME DE TOUS LES NOMBRES ALGÉBRIQUES RÉELS(1)

PAR

G. CANTOR

A HALLE a. S.

Written in 1874, Cantor's first major paper on the theory of sets, Über eine Eigenshaft des Inbegriffes aller reellen algebraischen Zahlen *(On a Property of the System of All the Real Algebraic Numbers), sparked a major controversy concerning the nature of infinite sets. To gather international support for his theory, Cantor had his papers translated into French. This 1883 French version of Cantor's work was published in the newly formed journal* Acta Mathematica. *Cantor's works were translated into English during the early twentieth century.*

SOLUTION

a. If sets have the same cardinal number, they are equivalent. Now, $n(A) = 4$ and $n(B) = 4$; therefore, $A \sim B$.

Because A and B are equivalent, their elements can be put into a one-to-one correspondence. One such correspondence follows:

John	Paul	George	Ringo
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
Lennon	McCartney	Harrison	Starr

b. Because $n(C) = 4$ and $n(D) = 3$, C and D are not equivalent.

c. A consists of all natural numbers from 1 to 50, inclusive. Hence, $n(A) = 50$. B consists of all odd natural numbers from 1 to 99, inclusive. Since half of the natural numbers from 1 to 100 are odd (and half are even), there are fifty ($100 \div 2 = 50$) odd natural numbers less than 100; that is, $n(B) = 50$. Because A and B have the same cardinal number, $A \sim B$.

Many different one-to-one correspondences may be established between the elements of A and B. One such correspondence follows:

$$\begin{array}{lllll} A = \{1, 2, 3, \ldots, & n, & \ldots, 48, 49, 50\} \\ \quad\ \ \updownarrow\ \updownarrow\ \updownarrow\ \ldots & \updownarrow & \ldots\ \updownarrow\ \ \updownarrow\ \ \updownarrow \\ B = \{1, 3, 5, \ldots, & (2n-1), & \ldots, 95, 97, 99\} \end{array}$$

That is, each natural number $n \in A$ is paired up with the odd number $(2n - 1) \in B$. The $n \leftrightarrow (2n - 1)$ part is crucial because it shows *each* individual correspondence. For example, it shows that $13 \in A$ corresponds to $25 \in B$ ($n = 13$, so $2n = 26$ and $2n - 1 = 25$). Likewise, $69 \in B$ corresponds to $35 \in A$ ($2n - 1 = 69$, so $2n = 70$ and $n = 35$).

As we have seen, if two sets have the same cardinal number, they are equivalent, and their elements can be put into a one-to-one correspondence. Conversely, if the elements of two sets can be put into a one-to-one correspondence, the sets have the same cardinal number and are equivalent. Intuitively, this result appears to be quite obvious. However, when Georg Cantor applied this relationship to infinite sets, he sparked one of the greatest philosophical debates of the nineteenth century.

Countable Sets

Consider the set of all counting numbers $N = \{1, 2, 3, \ldots\}$, which consists of an infinite number of elements. Each of these numbers is either odd or even. Defining O and E as $O = \{1, 3, 5, \ldots\}$ and $E = \{2, 4, 6, \ldots\}$, we have $O \cap E = \emptyset$ and $O \cup E = N$; the sets O and E are mutually exclusive, and their union forms the entire set of all counting numbers. Obviously, N contains elements that E does not. As we mentioned earlier, the fact that E is a *proper* subset of N might lead people to think that N is "bigger" than E. In fact, N and E are the "same size"; N and E each contain the same number of elements.

Recall that two sets are equivalent and have the same cardinal number if the elements of the sets can be matched up via a one-to-one correspondence. To show the existence of a one-to-one correspondence between the elements of two sets of numbers, we must find an explicit correspondence between the general elements of the two sets. In Example 1(c), we expressed the general correspondence as $n \leftrightarrow (2n - 1)$.

EXAMPLE 2

FINDING A ONE-TO-ONE CORRESPONDENCE BETWEEN TWO INFINITE SETS

a. Show that $E = \{2, 4, 6, 8, \ldots\}$ and $N = \{1, 2, 3, 4, \ldots\}$ are equivalent sets.
b. Find the element of N that corresponds to $1430 \in E$.
c. Find the element of N that corresponds to $x \in E$.

SOLUTION

a. To show that $E \sim N$, we must show that there exists a one-to-one correspondence between the elements of E and N. The elements of E and N can be paired up as follows:

$$\begin{array}{ll} N = \{1, 2, 3, 4, \ldots, & n, \ldots\} \\ \quad\ \ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \ldots & \updownarrow \\ E = \{2, 4, 6, 8, \ldots, & 2n, \ldots\} \end{array}$$

95057_02_ch02_p067-130.qxd 9/27/10 9:54 AM Page 121

Any natural number $n \in N$ corresponds with the even natural number $2n \in E$. Because there exists a one-to-one correspondence between the elements of E and N, the sets E and N are equivalent; that is, $E \sim N$.

b. $1430 = 2n \in E$, so $n = \frac{1430}{2} = 715 \in N$. Therefore, $715 \in N$ corresponds to $1430 \in E$.

c. $x = 2n \in E$, so $n = \frac{x}{2} \in N$. Therefore, $n = \frac{x}{2} \in N$ corresponds to $x = 2n \in E$.

We have just seen that the set of *even* natural numbers is equivalent to the set of *all* natural numbers. This equivalence implies that the two sets have the same number of elements! Although E is a proper subset of N, both sets have the same cardinal number; that is, $n(E) = n(N)$. Settling the controversy sparked by this seeming paradox, mathematicians today define a set to be an **infinite set** if it can be placed in a one-to-one correspondence with a proper subset of itself.

How many counting numbers are there? How many even counting numbers are there? We know that each set contains an infinite number of elements and that $n(N) = n(E)$, but how many is that? In the late nineteenth century, Georg Cantor defined the cardinal number of the set of counting numbers to be $\aleph_0$ (read "**aleph-null**"). Cantor utilized Hebrew letters, of which aleph, $\aleph$, is the first. Consequently, the proper response to "How many counting numbers are there?" is "There are aleph-null of them"; $n(N) = \aleph_0$. Any set that is equivalent to the set of counting numbers has cardinal number $\aleph_0$. A set is **countable** if it is finite or if it has cardinality $\aleph_0$.

Cantor was not the first to ponder the paradoxes of infinite sets. Hundreds of years before, Galileo had observed that part of an infinite set contained as many elements as the whole set. In his monumental *Dialogue Concerning the Two Chief World Systems* (1632), Galileo made a prophetic observation: "There are as many (perfect) squares as there are (natural) numbers because they are just as numerous as their roots." In other words, the elements of the sets $N = \{1, 2, 3, \ldots, n, \ldots\}$ and $S = \{1^2, 2^2, 3^2, \ldots, n^2, \ldots\}$ can be put into a one-to-one correspondence ($n \leftrightarrow n^2$). Galileo pondered which of the sets (perfect squares or natural numbers) was "larger" but abandoned the subject because he could find no practical application of this puzzle.

EXAMPLE 3

SHOWING THAT THE SET OF INTEGERS IS COUNTABLE Consider the following one-to-one correspondence between the set I of all integers and the set N of all natural numbers:

$$\begin{array}{rcccccl} N = \{ & 1, & 2, & 3, & 4, & 5, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ I = \{ & 0, & 1, & -1, & 2, & -2, & \ldots\} \end{array}$$

where an odd natural number n corresponds to a nonpositive integer $\frac{1-n}{2}$ and an even natural number n corresponds to a positive integer $\frac{n}{2}$.

a. Find the 613th integer; that is, find the element of I that corresponds to $613 \in N$.
b. Find the element of N that corresponds to $853 \in I$.
c. Find the element of N that corresponds to $-397 \in I$.
d. Find $n(I)$, the cardinal number of the set I of all integers.
e. Is the set of integers countable?

SOLUTION

a. $613 \in N$ is odd, so it corresponds to $\frac{1-613}{2} = \frac{-612}{2} = -306$. If you continued counting the integers as shown in the above correspondence, -306 would be the 613th integer in your count.

b. $853 \in I$ is positive, so

$$853 = \frac{n}{2} \qquad \textbf{multiplying by 2}$$
$$n = 1{,}706$$

$1{,}706 \in N$ corresponds to $853 \in I$.
This means that 853 is the 1,706th integer.

c. $-397 \in I$ is negative, so

$$-397 = \frac{1-n}{2}$$

$$-794 = 1 - n \quad \textbf{multiplying by 2}$$

$$-795 = -n \quad \textbf{subtracting 1}$$

$$n = 795 \quad \textbf{multiplying by } \mathbf{-1}$$

$795 \in N$ corresponds to $-397 \in I$.

This means that -397 is the 795th integer.

d. The given one-to-one correspondence shows that I and N have the same (infinite) number of elements; $n(I) = n(N)$. Because $n(N) = \aleph_0$, the cardinal number of the set of all integers is $n(I) = \aleph_0$.

e. By definition, a set is called countable if it is finite or if it has cardinality $\aleph_0$. The set of integers has cardinality $\aleph_0$, so it is countable. This means that we can "count off" all of the integers, as we did in parts (a), (b), and (c).

We have seen that the sets N (all counting numbers), E (all even counting numbers), and I (all integers) contain the same number of elements, $\aleph_0$. What about a set containing fractions?

EXAMPLE 4

DETERMINING WHETHER THE SET OF POSITIVE RATIONAL NUMBERS IS COUNTABLE Determine whether the set P of all positive rational numbers is countable.

SOLUTION

The elements of P can be systematically listed in a table of rows and columns as follows: All positive rational numbers whose denominator is 1 are listed in the first row, all positive rational numbers whose denominator is 2 are listed in the second row, and so on, as shown in Figure 2.43.

Each positive rational number will appear somewhere in the table. For instance, $\frac{125}{66}$ will be in row 66 and column 125. Note that not all the entries in Figure 2.43 are in lowest terms; for instance, $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}$, and so on are all equal to $\frac{1}{2}$.

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\cdots$
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\cdots$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

FIGURE 2.43 A list of all positive rational numbers.

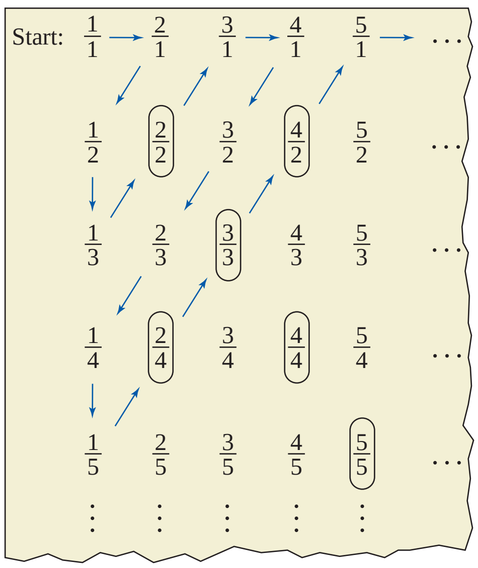


FIGURE 2.44 The circled rational numbers are not in lowest terms; they are omitted from the list.

Consequently, to avoid listing the same number more than once, an entry that is not in lowest terms must be eliminated from our list. To establish a one-to-one correspondence between P and N, we can create a zigzag diagonal pattern as shown by the arrows in Figure 2.44. Starting with $\frac{1}{1}$, we follow the arrows and omit any number that is not in lowest terms (the circled numbers in Figure 2.44). In this manner, a list of all positive rational numbers with no repetitions is created. Listing the elements of P in this order, we can put them in a one-to-one correspondence with N:

$$\begin{array}{rcccccccccccl} N = & \{1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ P = & \Big\{1, & 2, & \frac{1}{2}, & \frac{1}{3}, & 3, & 4, & \frac{3}{2}, & \frac{2}{3}, & \frac{1}{4}, & \frac{1}{5}, & 5, & \ldots\Big\} \end{array}$$

Any natural number n is paired up with the positive rational number found by counting through the "list" given in Figure 2.44. Conversely, any positive rational number is located somewhere in the list and is paired up with the counting number corresponding to its place in the list.

Therefore, $P \sim N$, so the set of all positive rational numbers is countable.

Uncountable Sets

Every infinite set that we have examined so far is countable; each can be put into a one-to-one correspondence with the set of all counting numbers and consequently has cardinality $\aleph_0$. Do not be misled into thinking that all infinite sets are countable! By utilizing a "proof by contradiction," Georg Cantor showed that the infinite set $A = \{x \mid 0 \le x < 1\}$ is *not* countable. This proof involves logic that is different from what you are used to. Do not let that intimidate you.

Assume that the set $A = \{x \mid 0 \le x < 1\}$ is countable; that is, assume that $n(A) = \aleph_0$. This assumption implies that the elements of A and N can be put into a one-to-one correspondence; each $a \in A$ can be listed and counted. Because the elements of A are nonnegative real numbers less than 1, each $a_n = 0.\square\square\square\square\square\ldots$. Say, for instance, the numbers in our list are

$a_1 = 0.3750000\ldots$ **the first element of A**

$a_2 = 0.7071067\ldots$ **the second element of A**

$a_3 = 0.5000000\ldots$ **the third element of A**

$a_4 = 0.6666666\ldots$ **and so on.**

The *assumption* that A is countable implies that every element of A appears somewhere in the above list. However, we can create an element of A (call it b) that is *not* in the list. We build b according to the "diagonal digits" of the numbers in our list and the following rule: If the digit "on the diagonal" is not zero, put a 0 in the corresponding place in b; if the digit "on the diagonal" is zero, put a 1 in the corresponding place in b.

The "diagonal digits" of the numbers in our list are as follows:

$a_1 = 0.\boxed{3}750000\ldots$

$a_2 = 0.7\boxed{0}71067\ldots$

$a_3 = 0.50\boxed{0}0000\ldots$

$a_4 = 0.666\boxed{6}666\ldots$

Because the first digit on the diagonal is 3, the first digit of b is 0. Because the second digit on the diagonal is 0, the second digit of b is 1. Using all the "diagonal digits" of the numbers in our list, we obtain $b = 0.0110\ldots$. Because $0 \le b < 1$, it follows that $b \in A$. However, the number b is not on our list of all elements of A. This is because

$b \neq a_1$ **(b and a_1 differ in the first decimal place)**
$b \neq a_2$ **(b and a_2 differ in the second decimal place)**
$b \neq a_3$ **(b and a_3 differ in the third decimal place), and so on**

This contradicts the assumption that the elements of A and N can be put into a one-to-one correspondence. Since the assumption leads to a contradiction, the assumption must be false; $A = \{x \mid 0 \le x < 1\}$ is not countable. Therefore, $n(A) \neq n(N)$. That is, A is an infinite set and $n(A) \neq \aleph_0$.

An infinite set that cannot be put into a one-to-one correspondence with N is said to be **uncountable.** Consequently, an uncountable set has *more* elements than the set of all counting numbers. This implies that there are different magnitudes of infinity! To distinguish the magnitude of A from that of N, Cantor denoted the cardinality of $A = \{x \mid 0 \le x < 1\}$ as $n(A) = c$ (c for **continuum**). Thus, Cantor showed that $\aleph_0 < c$. Cantor went on to show that A was equivalent to the entire set of all real numbers, that is, $A \sim \Re$. Therefore, $n(\Re) = c$.

Although he could not prove it, Cantor hypothesized that no set could have a cardinality between $\aleph_0$ and c. This famous unsolved problem, labeled the *Continuum Hypothesis*, baffled mathematicians throughout the first half of the twentieth century. It is said that Cantor suffered a devastating nervous breakdown in 1884 when he announced that he had a proof of the Continuum Hypothesis only to declare the next day that he could show the Continuum Hypothesis to be false!

The problem was finally "solved" in 1963. Paul J. Cohen demonstrated that the Continuum Hypothesis is independent of the entire framework of set theory; that is, it can be neither proved nor disproved by using the theorems of set theory. Thus, the Continuum Hypothesis is not provable.

Although no one has produced a set with cardinality between $\aleph_0$ and c, many sets with cardinality greater than c have been constructed. In fact, modern mathematicians have shown that there are *infinitely* many magnitudes of infinity! Using subscripts, these magnitudes, or cardinalities, are represented by $\aleph_0, \aleph_1, \aleph_2, \ldots$ and have the property that $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$. In this sense, the set N of all natural numbers forms the "smallest" infinite set. Using this subscripted notation, the Continuum Hypothesis implies that $c = \aleph_1$; that is, given that N forms the smallest infinite set, the set $\Re$ of all real numbers forms the next "larger" infinite set.

Points on a Line

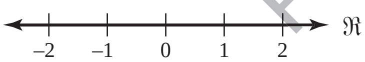


FIGURE 2.45
The real number line.

When students are first exposed to the concept of the real number system, a number line like the one in Figure 2.45 is inevitably introduced. The real number system, denoted by $\Re$, can be put into a one-to-one correspondence with all points on a line, such that every real number corresponds to exactly one point on a line and every point on a line corresponds to exactly one real number. Consequently, any (infinite) line contains c points. What about a line segment? For example, how many points does the segment [0, 1] contain? Does the segment [0, 2] contain twice as many points as the segment [0, 1]? Once again, intuition can lead to erroneous conclusions when people are dealing with infinite sets.

EXAMPLE 5

SHOWING THAT LINE SEGMENTS OF DIFFERENT LENGTHS ARE EQUIVALENT SETS OF POINTS Show that the line segments [0, 1] and [0, 2] are equivalent sets of points.

SOLUTION

Because the segment [0, 2] is twice as long as the segment [0, 1], intuition might tell us that it contains twice as many points. Not so! Recall that two sets are

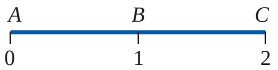


FIGURE 2.46 The interval [0, 2].

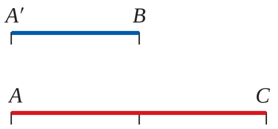


FIGURE 2.47 The intervals [*A*, *B*] and [*A*, *C*].

equivalent (and have the same cardinal number) if their elements can be put into a one-to-one correspondence.

On a number line, let *A* represent the point 0, let *B* represent 1, and let *C* represent 2, as shown in Figure 2.46. Our goal is to develop a one-to-one correspondence between the elements of the segments *AB* and *AC*. Now draw the segments separately, with *AB* above *AC*, as shown in Figure 2.47. (To distinguish the segments from each other, point *A* of segment *AB* has been relabeled as point *A*′.)

Extend segments *AA*′ and *CB* so that they meet at point *D*, as shown in Figure 2.48. Any point *E* on *A*′*B* can be paired up with the unique point *F* on *AC* formed by the intersection of lines *DE* and *AC*, as shown in Figure 2.49. Conversely, any point *F* on segment *AC* can be paired up with the unique point *E* on *A*′*B* formed by the intersection of lines *DF* and *A*′*B*. Therefore, a one-to-one correspondence exists between the two segments, so $[0, 1] \sim [0, 2]$. Consequently, the interval [0, 1] contains exactly the same number of points as the interval [0, 2]!

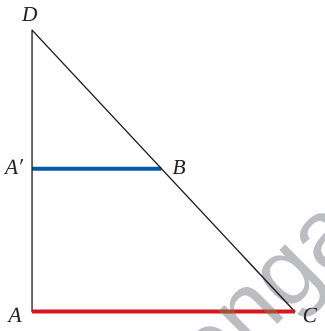


FIGURE 2.48 Extending *AA*′ and *CB* to form point *D*.

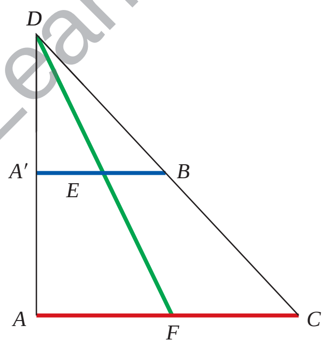


FIGURE 2.49 A one-to-one correspondence between line segments.

Even though the segment [0, 2] is twice as long as the segment [0, 1], each segment contains exactly the same number of points. The method used in Example 5 can be applied to any two line segments. Consequently, all line segments, regardless of their length, contain exactly the same number of points; a line segment 1 inch long has exactly the same number of points as a segment 1 mile long! Once again, it is easy to see why Cantor's work on the magnitude of infinity was so unsettling to many scholars.

Having concluded that all line segments contain the same number of points, we might ask how many points that is. What is the cardinal number? Given any line segment *AB*, it can be shown that $n(AB) = c$; the points of a line segment can be put into a one-to-one correspondence with the points of a line. Consequently, the interval [0, 1] contains the same number of elements as the entire real number system.

If things seem rather strange at this point, keep in mind that Cantor's pioneering work produced results that puzzled even Cantor himself. In a paper written in 1877, Cantor constructed a one-to-one correspondence between the points in a square (a two-dimensional figure) and the points on a line segment (a one-dimensional figure). Extending this concept, he concluded that a line segment and the entire two-dimensional plane contain exactly the same number of points, *c*. Communicating with his colleague Richard Dedekind, Cantor wrote, "I see it, but I do not believe it." Subsequent investigation has shown that the number of points contained in the interval [0, 1] is the same as the number of points contained in all of three-dimensional space! Needless to say, Cantor's work on the cardinality of infinity revolutionized the world of modern mathematics.

2.5 Exercises

In Exercises 1–10, find the cardinal numbers of the sets in each given pair to determine whether the sets are equivalent. If they are equivalent, list a one-to-one correspondence between their elements.

1. $S = \{$Sacramento, Lansing, Richmond, Topeka$\}$
$C = \{$California, Michigan, Virginia, Kansas$\}$

2. $T = \{$Wyoming, Ohio, Texas, Illinois, Colorado$\}$
$P = \{$Cheyenne, Columbus, Austin, Springfield, Denver$\}$

3. $R = \{a, b, c\}$; $G = \{\alpha, \beta, \chi, \delta\}$

4. $W = \{\text{I, II, III}\}$; $H = \{$one, two$\}$

5. $C = \{3, 6, 9, 12, \ldots, 63, 66\}$
$D = \{4, 8, 12, 16, \ldots, 84, 88\}$

6. $A = \{2, 4, 6, 8, \ldots, 108, 110\}$
$B = \{5, 10, 15, 20, \ldots, 270, 275\}$

7. $G = \{2, 4, 6, 8, \ldots, 498, 500\}$
$H = \{1, 3, 5, 7, \ldots, 499, 501\}$

8. $E = \{2, 4, 6, 8, \ldots, 498, 500\}$
$F = \{3, 6, 9, 12, \ldots, 750, 753\}$

9. $A = \{1, 3, 5, \ldots, 121, 123\}$
$B = \{125, 127, 129, \ldots, 245, 247\}$

10. $S = \{4, 6, 8, \ldots, 664, 666\}$
$T = \{5, 6, 7, \ldots, 335, 336\}$

11. **a.** Show that the set O of all odd counting numbers, $O = \{1, 3, 5, 7, \ldots\}$, and $N = \{1, 2, 3, 4, \ldots\}$ are equivalent sets.
b. Find the element of N that corresponds to $1{,}835 \in O$.
c. Find the element of N that corresponds to $x \in O$.
d. Find the element of O that corresponds to $782 \in N$.
e. Find the element of O that corresponds to $n \in N$.

12. **a.** Show that the set W of all whole numbers, $W = \{0, 1, 2, 3, \ldots\}$, and $N = \{1, 2, 3, 4, \ldots\}$ are equivalent sets.
b. Find the element of N that corresponds to $932 \in W$.
c. Find the element of N that corresponds to $x \in W$.
d. Find the element of W that corresponds to $932 \in N$.
e. Find the element of W that corresponds to $n \in N$.

13. **a.** Show that the set T of all multiples of 3, $T = \{3, 6, 9, 12, \ldots\}$, and $N = \{1, 2, 3, 4, \ldots\}$ are equivalent sets.
b. Find the element of N that corresponds to $936 \in T$.
c. Find the element of N that corresponds to $x \in T$.
d. Find the element of T that corresponds to $936 \in N$.
e. Find the element of T that corresponds to $n \in N$.

14. **a.** Show that the set F of all multiples of 5, $F = \{5, 10, 15, 20, \ldots\}$, and $N = \{1, 2, 3, 4, \ldots\}$ are equivalent sets.
b. Find the element of N that corresponds to $605 \in F$.
c. Find the element of N that corresponds to $x \in F$.
d. Find the element of F that corresponds to $605 \in N$.
e. Find the element of F that corresponds to $n \in N$.

15. Consider the following one-to-one correspondence between the set A of all even integers and the set N of all natural numbers:

$$\begin{array}{rcccccl} N = \{ & 1, & 2, & 3, & 4, & 5, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A = \{ & 0, & 2, & -2, & 4, & -4, & \ldots\} \end{array}$$

where an odd natural number n corresponds to the nonpositive integer $1 - n$ and an even natural number n corresponds to the positive even integer n.
a. Find the 345th even integer; that is, find the element of A that corresponds to $345 \in N$.
b. Find the element of N that corresponds to $248 \in A$.
c. Find the element of N that corresponds to $-754 \in A$.
d. Find $n(A)$.

16. Consider the following one-to-one correspondence between the set B of all odd integers and the set N of all natural numbers:

$$\begin{array}{rcccccl} N = \{ & 1, & 2, & 3, & 4, & 5, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ B = \{ & 1, & -1, & 3, & -3, & 5, & \ldots\} \end{array}$$

where an even natural number n corresponds to the negative odd integer $1 - n$ and an odd natural number n corresponds to the odd integer n.
a. Find the 345th odd integer; that is, find the element of B that corresponds to $345 \in N$.
b. Find the element of N that corresponds to $241 \in B$.
c. Find the element of N that corresponds to $-759 \in B$.
d. Find $n(B)$.

In Exercises 17–22, show that the given sets of points are equivalent by establishing a one-to-one correspondence.

17. the line segments $[0, 1]$ and $[0, 3]$

18. the line segments $[1, 2]$ and $[0, 3]$

19. the circle and square shown in Figure 2.50

HINT: Draw one figure inside the other.

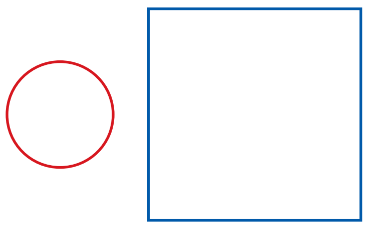


FIGURE 2.50

20. the rectangle and triangle shown in Figure 2.51

HINT: Draw one figure inside the other.

FIGURE 2.51

21. a circle of radius 1 cm and a circle of 5 cm

HINT: Draw one figure inside the other.

22. a square of side 1 cm and a square of side 5 cm

HINT: Draw one figure inside the other.

23. Show that the set of all real numbers between 0 and 1 has the same cardinality as the set of all real numbers.

HINT: Draw a semicircle to represent the set of real numbers between 0 and 1 and a line to represent the set of all real numbers, and use the method of Example 5.

Answer the following questions using complete sentences and your own words.

• CONCEPT QUESTIONS

24. What is the cardinal number of the "smallest" infinite set? What set or sets have this cardinal number?

• HISTORY QUESTIONS

25. What aspect of Georg Cantor's set theory caused controversy among mathematicians and philosophers?

26. What contributed to Cantor's breakdown in 1884?

27. Who demonstrated that the Continuum Hypothesis cannot be proven? When?

WEB PROJECT

28. Write a research paper on any historical topic referred to in this section. Following is a partial list of topics:

- Georg Cantor
- Richard Dedekind
- Paul J. Cohen
- Bernhard Bolzano
- Leopold Kronecker
- the Continuum Hypothesis

Some useful links for this web project are listed on the text web site: **www.cengage.com/math/johnson**

2 CHAPTER REVIEW

TERMS

aleph-null
cardinal number
combination
combinatorics
complement
continuum
countable set
De Morgan's Laws
distinguishable permutations
element
empty set
equal sets
equivalent sets
factorial
Fundamental Principle of Counting
improper subset
infinite set
intersection
mutually exclusive
one-to-one correspondence
permutation
proper subset
roster notation
set
set-builder notation
set theory
subset
tree diagram
uncountable set
union
universal set
Venn diagram
well-defined set

Review Exercises

1. State whether the given set is well-defined.

a. the set of all multiples of 5.

b. the set of all difficult math problems

c. the set of all great movies

d. the set of all Oscar-winning movies

2. Given the sets

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A = \{0, 2, 4, 6, 8\}$

$B = \{1, 3, 5, 7, 9\}$

find the following using the roster method.

a. A' **b.** B'

c. $A \cup B$ **d.** $A \cap B$

3. Given the sets $A = \{$Maria, Nobuko, Leroy, Mickey, Kelly$\}$ and $B = \{$Rachel, Leroy, Deanna, Mickey$\}$, find the following.

a. $A \cup B$ **b.** $A \cap B$

4. List all subsets of $C = \{$Dallas, Chicago, Tampa$\}$. Identify which subsets are proper and which are improper.

5. Given $n(U) = 61$, $n(A) = 32$, $n(B) = 26$, and $n(A \cup B) = 40$, do the following.

a. Find $n(A \cap B)$.

b. Draw a Venn diagram illustrating the composition of U.

In Exercises 6 and 7, use a Venn diagram like the one in Figure 2.15 to shade in the region corresponding to the indicated set.

6. $(A' \cup B)'$

7. $(A \cap B')'$

In Exercises 8 and 9, use a Venn diagram like the one in Figure 2.36 to shade in the region corresponding to the indicated set.

8. $A' \cap (B \cup C')$

9. $(A' \cap B) \cup C'$

10. A survey of 1,000 college seniors yielded the following information: 396 seniors favored capital punishment, 531 favored stricter gun control, and 237 favored both.

a. How many favored capital punishment or stricter gun control?

b. How many favored capital punishment but not stricter gun control?

c. How many favored stricter gun control but not capital punishment?

d. How many favored neither capital punishment nor stricter gun control?

11. A survey of recent college graduates yielded the following information: 70 graduates earned a degree in mathematics, 115 earned a degree in education, 23 earned degrees in both mathematics and education, and 358 earned a degree in neither mathematics nor education.

a. What percent of the college graduates earned a degree in mathematics or education?

b. What percent of the college graduates earned a degree in mathematics only?

c. What percent of the college graduates earned a degree in education only?

12. A local anime fan club surveyed 67 of its members regarding their viewing habits last weekend, and the following information was obtained: 30 members watched an episode of *Naruto*, 44 watched an episode of *Death Note*, 23 watched an episode of *Inuyasha*, 20 watched both *Naruto* and *Inuyasha*, 5 watched *Naruto* and *Inuyasha* but not *Death Note*, 15 watched both *Death Note* and *Inuyasha*, and 23 watched only *Death Note*.

a. How many of the club members watched exactly one of the shows?

b. How many of the club members watched all three shows?

c. How many of the club members watched none of the three shows?

13. An exit poll yielded the following information concerning people's voting patterns on Propositions A, B, and C: 305 people voted yes on A, 95 voted yes only on A, 393 voted yes on B, 192 voted yes only on B, 510 voted yes on A or B, 163 voted yes on C, 87 voted yes on all three, and 249 voted no on all three. What percent of the voters voted yes on more than one proposition?

14. Given the sets $U = \{$a, b, c, d, e, f, g, h, i$\}$, $A = \{$b, d, f, g$\}$, and $B = \{$a, c, d, g, i$\}$, use De Morgan's Laws to find the following.

a. $(A' \cup B)'$ **b.** $(A \cap B')'$

15. Refer to the Venn diagram depicted in Figure 2.32.

a. In a group of 100 Americans, how many have type O or type A blood?

b. In a group of 100 Americans, how many have type O and type A blood?

c. In a group of 100 Americans, how many have neither type O nor type A blood?

16. Refer to the Venn diagram depicted in Figure 2.28.

a. For a typical group of 100 Americans, fill in the cardinal number of each region in the diagram.

b. In a group of 100 Americans, how many have type O blood or are Rh+?

c. In a group of 100 Americans, how many have type O blood and are Rh+?

d. In a group of 100 Americans, how many have neither type O blood nor are Rh+?

17. Sid and Nancy are planning their anniversary celebration, which will include viewing an art exhibit, having dinner, and going dancing. They will go either to the Museum of Modern Art or to the New Photo Gallery; dine either at Stars, at Johnny's, or at the Chelsea; and go dancing either at Le Club or at Lizards.
    a. In how many different ways can Sid and Nancy celebrate their anniversary?
    b. Construct a tree diagram to list all possible ways in which Sid and Nancy can celebrate their anniversary.

18. A certain model of pickup truck is available in five exterior colors, three interior colors, and three interior styles. In addition, the transmission can be either manual or automatic, and the truck can have either two-wheel or four-wheel drive. How many different versions of the pickup truck can be ordered?

19. Each student at State University has a student I.D. number consisting of five digits (the first digit is nonzero, and digits can be repeated) followed by two of the letters A, B, C, and D (letters cannot be repeated). How many different student numbers are possible?

20. Find the value of each of the following.
    a. $(17 - 7)!$ **b.** $(17 - 17)!$
    c. $\dfrac{82!}{79!}$ **d.** $\dfrac{27!}{20!7!}$

21. In how many ways can you select three out of eleven items under the following conditions?
    a. Order of selection is not important.
    b. Order of selection is important.

22. Find the value of each of the following.
    a. ${}_{15}P_4$ **b.** ${}_{15}C_4$ **c.** ${}_{15}P_{11}$

23. A group of ten women and twelve men must select a three-person committee. How many committees are possible if it must consist of the following?
    a. one woman and two men
    b. any mixture of men and women
    c. a majority of men

24. A volleyball league has ten teams. If every team must play every other team once in the first round of league play, how many games must be scheduled?

25. A volleyball league has ten teams. How many different end-of-the-season rankings of first, second, and third place are possible (disregarding ties)?

26. Using a standard deck of fifty-two cards (no jokers), how many seven-card poker hands are possible?

27. Using a standard deck of fifty-two cards and two jokers, how many seven-card poker hands are possible?

28. A 6/42 lottery requires choosing six of the numbers 1 through 42. How many different lottery tickets can you choose?

In Exercises 29 and 30, find the number of permutations of the letters in each word.

29. **a.** FLORIDA **b.** ARIZONA **c.** MONTANA

30. **a.** AFFECT **b.** EFFECT

31. What is the major difference between permutations and combinations?

32. Use Pascal's Triangle to answer the following.
    a. In which entry in which row would you find the value of ${}_7C_3$?
    b. In which entry in which row would you find the value of ${}_7C_4$?
    c. How is the value of ${}_7C_3$ related to the value of ${}_7C_4$? Why?
    d. What is the location of ${}_nC_r$? Why?

33. Given the set $S = \{a, b, c\}$, answer the following.
    a. How many one-element subsets does S have?
    b. How many two-element subsets does S have?
    c. How many three-element subsets does S have?
    d. How many zero-element subsets does S have?
    e. How many subsets does S have?
    f. How is the answer to part (e) related to $n(S)$?

In Exercises 34–36, find the cardinal numbers of the sets in each given pair to determine whether the sets are equivalent. If they are equivalent, list a one-to-one correspondence between their elements.

34. $A = \{\text{I, II, III, IV, V}\}$ and $B = \{\text{one, two, three, four, five}\}$

35. $C = \{3, 5, 7, \ldots, 899, 901\}$ and $D = \{2, 4, 6, \ldots, 898, 900\}$

36. $E = \{\text{Ronald}\}$ and $F = \{\text{Reagan, McDonald}\}$

37. **a.** Show that the set S of perfect squares, $S = \{1, 4, 9, 16, \ldots\}$, and $N = \{1, 2, 3, 4, \ldots\}$ are equivalent sets.
    b. Find the element of N that corresponds to $841 \in S$.
    c. Find the element of N that corresponds to $x \in S$.
    d. Find the element of S that corresponds to $144 \in N$.
    e. Find the element of S that corresponds to $n \in N$.

38. Consider the following one-to-one correspondence between the set A of all integer multiples of 3 and the set N of all natural numbers:

$$\begin{array}{l}
N = \{1,\ 2,\ \ 3,\ 4,\ \ 5, \ldots\} \\
\quad\ \ \ \updownarrow\ \updownarrow\ \ \updownarrow\ \updownarrow\ \ \updownarrow \\
A = \{0, 3, -3, 6, -6, \ldots\}
\end{array}$$

    where an odd natural number n corresponds to the nonpositive integer $\frac{3}{2}(1 - n)$, and an even natural number n corresponds to the positive even integer $\frac{3}{2}n$.
    a. Find the element of A that corresponds to $396 \in N$.
    b. Find the element of N that corresponds to $396 \in A$.
    c. Find the element of N that corresponds to $-153 \in A$.
    d. Find $n(A)$.

39. Show that the line segments [0, 1] and [0, π] are equivalent sets of points by establishing a one-to-one correspondence.

Answer the following questions using complete sentences and your own words.

• CONCEPT QUESTIONS

40. What is the difference between proper and improper subsets?

41. Explain the difference between {0} and $\varnothing$.

42. What is a factorial?

43. What is the difference between permutations and combinations?

• HISTORY QUESTIONS

44. What roles did the following people play in the development of set theory and combinatorics?

- Georg Cantor
- Augustus De Morgan
- Christian Kramp
- Chu Shih-chieh
- John Venn

THE NEXT LEVEL

If a person wants to pursue an advanced degree (something beyond a bachelor's or four-year degree), chances are the person must take a standardized exam to gain admission to a school or to be admitted into a specific program. These exams are intended to measure verbal, quantitative, and analytical skills that have developed throughout a person's life. Many classes and study guides are available to help people prepare for the exams. The following questions are typical of those found in the study guides.

Exercises 45–48 refer to the following: Two doctors in a local clinic are determining which days of the week they will be on call. Each day, Monday through Sunday, is to be assigned to one of two doctors, A and B, such that the assignment is consistent with the following conditions:

- No day is assigned to both doctors.
- Neither doctor has more than four days.
- Monday and Thursday must be assigned to the same doctor.
- If Tuesday is assigned to doctor *A*, then so is Sunday.
- If Saturday is assigned to doctor *B*, then Friday is not assigned to doctor *B*.

45. Which one of the following could be a complete and accurate list of the days assigned to doctor *A*?

a. Monday, Thursday
b. Monday, Tuesday, Sunday
c. Monday, Thursday, Sunday
d. Monday, Tuesday, Thursday
e. Monday, Thursday, Friday, Sunday

46. Which of the following cannot be true?

a. Thursday and Sunday are assigned to doctor *A*.
b. Friday and Saturday are assigned to doctor *A*.
c. Monday and Tuesday are assigned to doctor *B*.
d. Monday, Wednesday, and Sunday are assigned to doctor *A*.
e. Tuesday, Wednesday, and Saturday are assigned to doctor *B*.

47. If Friday and Sunday are both assigned to doctor *B*, how many different ways are there to assign the other five days to the doctors?

a. 1 **b.** 2 **c.** 3
d. 5 **e.** 6

48. If doctor *A* has four days, none of which is Monday, which of the following days must be assigned to doctor *A*?

a. Tuesday **b.** Wednesday **c.** Friday
d. Saturday **e.** Sunday